Practice Problem Set 2 - Riemann-Stieltjes integration

More problems may be added to this set. Stay tuned.

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1. If $\mathcal{R}_\alpha[a,b]$ contains all step functions on $[a,b]$, show that α is continuous.

2. Given a sequence $\{x_n\}$ of distinct points in (a,b) and a sequence $\{c_n\}$ of positive numbers with $\sum_{n=1}^{\infty} c_n < \infty$, define an increasing function α on $[a,b]$ by setting
$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad \text{where } I(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0. \end{cases}$$
Make sure that α is well-defined and non-decreasing. Show that
$$\int f\, d\alpha = \sum_{n=1}^{\infty} c_n f(x_n) \quad \text{for every continuous function } f \text{ on } [a,b].$$

3. Construct a nonconstant nondecreasing function α and a nonzero continuous function $f \in \mathcal{R}_\alpha[a,b]$ such that $\int_a^b |f| d\alpha = 0$. Is it possible to choose α to also be continuous? Explain.

4. Show that
$$||f||_1 = \int_a^b |f(x)|\, dx$$
defines a norm on $\mathcal{C}[a,b]$. Compare this norm with the sup norm $||\cdot||_\infty$ on $\mathcal{C}[a,b]$ that we already know. Does $||\cdot||_1$ define a norm on all of $\mathcal{R}[a,b]$? Explain.

5. Give an example of a sequence of Riemann integrable functions on $[0,1]$ that converges pointwise to a nonintegrable function.

6. Let α be continuous and nondecreasing. Given $f \in \mathcal{R}_\alpha[a,b]$ and $\epsilon > 0$, prove the following:
   (a) There exists a step function h on $[a,b]$ with $||h||_\infty \leq ||f||_\infty$ such that $\int_a^b |f - h|\, d\alpha < \epsilon$.
   (b) There exists a continuous function g on $[a,b]$ with $||g||_\infty \leq ||f||_\infty$ such that $\int_a^b |f - g|\, d\alpha < \epsilon$.

   Thus a Riemann-Stieltjes integrable function can be "approximated", in the sense above, by step functions and continuous functions.

7. Show that $\mathrm{BV}[a,b]$ is closed under the norm $||\cdot||_{\mathrm{BV}}$.

8. Suppose that α' exists and is a bounded Riemann integrable function on $[a,b]$. Given any bounded function f on $[a,b]$, show that $f \in \mathcal{R}_\alpha[a,b]$ if and only if $f\alpha' \in \mathcal{R}[a,b]$. In either case,
$$\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx.$$

9. Given a sequence of scalars $\{c_n\}$ and a sequence of distinct points $\{x_n\}$ in (a, b), define $f(x) = c_n$ if $x = x_n$ for some n, and $f(x) = 0$ otherwise. Under what conditions is f of bounded variation on $[a, b]$?

10. Let $I(x) = 0$ if $x < 0$ and $I(x) = 1$ if $x \geq 0$. Given a sequence of scalars $\{c_n\}$ with $\sum_{n=1}^{\infty} |c_n| < \infty$ and a sequence of distinct points $\{x_n\}$ in $(a, b]$, define $f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$ for $x \in [a, b]$. Show that $f \in \mathrm{BV}[a, b]$ and that $V_a^b f = \sum_{n=1}^{\infty} |c_n|$.

11. If $\alpha \in \mathrm{BV}[a, b]$, show that $\mathcal{R}_\alpha[a, b] \cap \mathcal{B}[a, b]$ is a closed subspace of $\mathcal{B}[a, b]$.

12. Let α be a non-decreasing function on $[a, b]$, and let $f \in \mathcal{R}_\alpha[a, b]$. Define $F(x) = \int_a^x f \, d\alpha$ for $a \leq x \leq b$. Show that
    (a) $F \in \mathrm{BV}[a, b]$;
    (b) F is continuous at each point where α is continuous;
    (c) F is differentiable at each point where α is differentiable and f is continuous. At any such point, $F'(x) = f(x)\alpha'(x)$.

13. Show that $||f_1 f_2||_{\mathrm{BV}} \leq ||f_1||_{\mathrm{BV}} ||f_2||_{\mathrm{BV}}$. (*Hint:* Use the decomposition $f = p - n + f(a)$ where p and n denote the positive and negative variation functions of f respectively.)