Midterm Solutions - Math 321

1. *Give complete definitions of the following terms:*

   (a) *an equicontinuous family of functions in* $\mathcal{C}[0,1]$.

   Solution. A family of functions $\mathcal{F} \subseteq \mathcal{C}[0,1]$ is said to be equicontinuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x - y| < \delta,\ x, y \in [0,1] \text{ and for all } f \in \mathcal{F}. \tag{1}$$

□

   (b) *a sublattice of* $\mathcal{B}[0,1]$.

   Solution. The space $\mathcal{B}[0,1]$ is equipped with a partial order $\leq$ defined as follows: given $f, g \in \mathcal{B}[0,1]$, we say that $f \leq g$ if for every $x \in [0,1]$, the real number $f(x)$ is less than or equal to the real number $g(x)$. Given $f, g \in \mathcal{B}[0,1]$, we define functions $m, M \in \mathcal{B}[0,1]$ as follows:

$$m(x) = \min\big[f(x), g(x)\big], \qquad M(x) = \max\big[f(x), g(x)\big].$$

   These functions have the following properties: if $h, k \in \mathcal{B}[0,1]$ are such that $f, g \leq h$ and $k \leq f, g$, then we must have that $k \leq m$ and $M \leq h$. We refer to m and M as $\min(f,g)$ and $\max(f,g)$ respectively.

   We say that $\mathcal{L} \subseteq \mathcal{B}[0,1]$ is a sublattice if for every $f, g \in \mathcal{L}$, the functions $\min(f,g)$ and $\max(f,g)$ also lie in $\mathcal{L}$. □

2. *Give examples of the following:*

   (a) *Function classes* $\mathcal{F}, \mathcal{G} \subseteq \mathcal{C}[0,1]$ *such that both* $\mathcal{F}$ *and* $\mathcal{G}$ *consist of infinitely many non-constant functions and are uniformly bounded, for which* $\mathcal{F}$ *is equicontinuous but* $\mathcal{G}$ *is not.*

   Solution. Consider the following subsets of $\mathcal{C}[0,1]$:

$$\mathcal{F} = \{f_n(x) = x + \frac{1}{n} : n \in \mathbb{N}\}, \qquad \mathcal{G} = \{g_n(x) = x^n : n \in \mathbb{N}\}.$$

   Both collections are uniformly bounded, $\mathcal{F}$ by 2 and $\mathcal{G}$ by 1.
   The collection $\mathcal{F}$ is equicontinuous, since (1) holds with $\delta = \epsilon$. However $\mathcal{G}$ is not equicontinuous. We can see this in two ways. Consider $x_0 = 1$. Aiming for a contradiction, suppose for any $\epsilon > 0$ there is a $\delta > 0$ such that $|x - 1| < \delta$ implies $|x^n - 1| < \epsilon$ for all n. Then choosing $x = 1 - \frac{1}{n}$ for sufficiently large $n > 1/\delta$, we find that $|(1 - (1 - \frac{1}{n})^n| < \epsilon$. But the left hand side of this inequality converges to e^{-1} as $n \to \infty$, and hence this inequality is false for large n if $\epsilon = \frac{e-1}{2e}$ for example.
   Alternatively, observe that $g_n(x)$ converges pointwise to

$$g(x) = \begin{cases} 0 & \text{if } \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

   which is not continuous and hence the convergence cannot be uniform and by HW5 Q5 $\mathcal{G}$ cannot be equicontinuous.

□

(b) *A nontrivial sublattice of $\mathcal{C}[0,1]$ that is a subspace but not a subalgebra. Here "nontrivial" means that the sublattice must contain at least one non-constant function.*

Solution. The class of piecewise linear functions on $\mathcal{C}[0,1]$ is a subspace but not an algebra. □

3. *Given any function $f \in \mathcal{C}(\mathbb{R}^n)$, show that there exists a sequence $\{p_k\}$ of polynomials in n variables that converges uniformly to f on every compact subset of $\mathbb{R}^n$.* (

Solution. For any $N \geq 1$, the class of polynomials of n variables forms dense subalgebra of $\mathcal{C}(B_N)$ where $B_N = \{x \in \mathbb{R}^n : \|x\| \leq N\}$. This follows from the Stone-Weierstrass theorem, since this subalgebra vanishes nowhere (due to the presence of the constant function 1) and separates points (due to the presence of the polynomials $f_j(x_1, \cdots, x_n) = x_j$). Thus there exists a polynomial p_N such that $\sup_{x \in B_N} |f(x) - p_N(x)| < \frac{1}{N}$.

We claim that the sequence $\{p_N\}$ converges uniformly on every compact subset of $\mathbb{R}^n$. Indeed, given any compact set K and $\epsilon > 0$, we find a large enough N such that $K \subseteq B_N$ and $N > \frac{1}{\epsilon}$. Then for all $k \geq N$,

$$\sup_{x \in K} |p_k(x) - f(x)| \leq \sup_{x \in B_k} |p_k(x) - f(x)| < \frac{1}{k} \leq \frac{1}{N} < \epsilon,$$

which proves the result. □

4. *Give brief answers to the following questions. The answer should be in the form of a short proof or an example, as appropriate.*

(a) *Is it true that any continuous function f in $\mathcal{C}[1,2]$ can be uniformly approximated by a sequence of even polynomials, and also by a sequence of odd polynomials?*

Solution. Yes. The function $f(\sqrt{x})$ is continuous on $[1,2]$, hence by Weierstrass's approximation theorem there is a polynomial p that is approximates it arbitrarily closely in sup norm. This implies that f is approximated by the even polynomial $p(x^2)$. In order to approximate f by an odd polynomial, note that the continuous function $f(x)/x$ can be approximated by an even polynomial by the first part of this problem. □

(b) *Would your answer to part (a) change if f lies in $\mathcal{C}[-1,2]$?*

Solution. The answer would change, and the statement of part (a) is no longer true. Since even polynomials do not separate the points x and $-x$, a function such as $\sin x$ would not be approximable by even polynomials. On the other hand, odd polynomials must vanish at zero, hence $\cos x$ cannot be approximated by odd polynomials. □

(c) *Let $\{f_n : n \geq 1\}$ be a sequence in $\mathcal{C}[a,b]$ with $||f_n||_\infty \leq 1$ for all n. Define*

$$F_n(x) = \int_a^x f_n(t)\, dt.$$

Does $\{F_n\}$ have a uniformly convergent subsequence?

Solution. Yes. The set $\{F_n\}$ is uniformly bounded by $(b-a)$ (using the bound on $||f_n||_\infty$) and consists of Lipschitz functions with Lipschitz constant 1,

$$|F_n(x) - F_n(y)| = \Big|\int_x^y f_n(t)\,dt\Big| \le |x-y|$$

hence equicontinuous. By the Arzela-Ascoli theorem, this collection of functions is relatively compact in $\mathcal{C}[a,b]$, hence admits a uniformly convergent subsequence. □

(d) *Can a sequence of Riemann integrable functions on* $[a,b]$ *converge pointwise to a non-integrable function?*

Solution. Yes. Let $\mathbb{Q} = \{q_1, q_2, \cdots, q_n, \cdots\}$, and set $f_n = \chi_{\{q_1,\cdots,q_n\}}$. Each f_n is Riemann-integrable (with integral 0), and converges pointwise to $\chi_{\mathbb{Q}}$ which is not. □

(e) *Consider the space* $\mathcal{C}_0(\mathbb{R})$ *of all continuous functions "vanishing at infinity"*

$$\mathcal{C}_0(\mathbb{R}) = \Big\{f \in \mathcal{C}(\mathbb{R}) \,:\, \lim_{|x|\to\infty} f(x) = 0\Big\},$$

endowed with the sup norm. Is this space separable?

Solution. Yes, the space is separable. For each $N \ge 1$, let $\mathcal{G}_N$ denote the space of polygonal functions on $[-N, N]$ with rational nodes which vanish on $|x| \ge N$. We have proved in class that each $\mathcal{G}_N$ is countable. Further given any $g \in \mathcal{C}[-N, N]$ such that $|g(\pm N)| < \kappa$, we can use the usual uniform continuity argument to find $G \in \mathcal{G}_N$ such that $\sup_{|x|\le N} |g(x) - G(x)| < \kappa$.
We claim that the countable union of these sets $\mathcal{G} = \bigcup_{N=1}^\infty \mathcal{G}_N$, which is countable, is dense in $\mathcal{C}_0(\mathbb{R})$. To see this fix $\epsilon > 0$ and any $f \in \mathcal{C}_0(\mathbb{R})$. Now choose N such that $|f(x)| < \epsilon$ for $|x| \ge N$. Then pick $G \in \mathcal{G}_N$ such that G approximates f restricted to $[-N, N]$ in the sense described above, i.e., $\sup_{|x|\le N} |f(x) - G(x)| < \epsilon$. Combining the two inequalities we get that $||f - G||_\infty = \max[\sup_{|x|\le N} |f(x) - G(x)|, \sup_{|x|\ge N} |f(x)|] < \epsilon$, as desired. □