Math 300 Homework 3 Solution

1. (Section 2.2 Q7)

   (c)
$$-1+\frac{i}{n}=\sqrt{1+\frac{1}{n^2}}e^{i(\pi-\tan^{-1}(n^{-1}))}$$
$$\lim_{n\to\infty} z_n=\lim_{n\to\infty}\left(\pi-\tan^{-1}(n^{-1})\right)=\pi$$

   (e)
$$\lim_{n\to\infty}\left|\left(\frac{1-i}{4}\right)^n\right|=\lim_{n\to\infty}\left(\frac{1}{2\sqrt{2}}\right)^n=0$$
$$\therefore \lim_{n\to\infty} z_n=0$$

   (f) Note that for all $k\geq 1$, we have
$$z_{5k}=\exp(2k\pi i)=1,$$
$$z_{5k+1}=\exp\left(2k\pi i+\frac{2\pi i}{5}\right)=\exp\left(\frac{2\pi i}{5}\right)\neq 1.$$

   Hence, the sequence z_n diverges.

2. (Section 2.2 Q11)

   (c)
$$\lim_{z\to 3i}\frac{z^2+9}{z-3i}=\lim_{z\to 3i}\frac{(z+3i)(z-3i)}{z-3i}=\lim_{z\to 3i} z+3i=6i$$

   (d)
$$\lim_{z\to i}\frac{z^2+i}{z^4-1}=\infty$$

3. (Section 2.3 Q4)

   (c) Case 1, $z=0$.
$$\lim_{\Delta z\to 0}\frac{|0+\Delta z|-|0|}{\Delta z}=\lim_{\Delta z\to 0}\frac{\sqrt{(\Delta z)^2+(\Delta y)^2}}{\Delta x+i\Delta y}$$
$$=\begin{cases}\pm 1, & \text{if } \Delta z=\Delta x\\ -i, & \text{if } \Delta z=\pm i\Delta y\end{cases}$$

Case 2, $z \neq 0$.

$$
\begin{aligned}
&\lim_{\Delta z \to 0} \frac{|z+\Delta z| - |z|}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\sqrt{(x+\Delta x)^2 + (y+\Delta y)^2} - \sqrt{x^2+y^2}}{\Delta x + i\Delta y} \\
&= \lim_{\Delta z \to 0} \frac{(x+\Delta x)^2 + (y+\Delta y)^2 - (x^2+y^2)}{(\Delta x + i\Delta y)(\sqrt{(x+\Delta x)^2 + (y+\Delta y)^2} + \sqrt{x^2+y^2})} \\
&= \lim_{\Delta z \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2y\Delta y + (\Delta y)^2}{(\Delta x + i\Delta y)(\sqrt{(x+\Delta x)^2 + (y+\Delta y)^2} + \sqrt{x^2+y^2})} \\
&= \begin{cases} \dfrac{x}{\sqrt{x^2+y^2}}, & \text{if } \Delta z = \Delta x,\ z \neq 0 \\ \dfrac{y}{i\sqrt{x^2+y^2}}, & \text{if } \Delta z = i\Delta y,\ z \neq 0 \end{cases}
\end{aligned}
$$

4. (Section 2.3 Q11)

   (b) Nowhere analytic.

   (f) Note that $f(z) = z + \frac{1}{z}$, f is analytic except at $z = 0$.

5. (Section 2.3 Q13)

   (c) Take $f(z) = 1$, $g(z) = z$. $\frac{1}{z}$ is not differentiable at $z = 0$. So $\frac{f(z)}{g(z)}$ is not always entire.

   (e) Take $f(z) = z$. $\frac{1}{z}$ is not differentiable at $z = 0$. So $f\left(\frac{1}{z}\right)$ is not always entire.

   (f) Since the composition of two entire functions is entire, $g(z^2+2)$ is always entire.