Solutions to MATH 300 Homework 2

EXERCISES 1.5

4. **(a)** For $z = \sqrt{3} - i$, $r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$, $\theta = \arctan(-1/\sqrt{3}) = -\pi/6$ (since $z = \sqrt{3} - i$ is in the 4-th quadrant). By using the identity $z^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$, we have

$$\begin{aligned}(\sqrt{3} - i)^7 &= 2^7 e^{-7\pi i/6} = 128(\cos(-7\pi/6) + i \sin(-7\pi/6)) \\ &= 128(-\sqrt{3}/2 + i/2)) \\ &= -64\sqrt{3} + i\ 64.\end{aligned}$$

(b) For $z = 1+i$, $r = \sqrt{1^2 + 1^2} = 2$, $\theta = \arctan(1) = \pi/4$ (since $z = 1+i$ is in the first quadrant). By using the identity $z^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$, we have

$$\begin{aligned}(1+i)^{95} &= (\sqrt{2})^{95} e^{95\pi i/4} = (\sqrt{2})^{95}(\cos(95\pi/4) + i \sin(95\pi/4)) \\ &= (\sqrt{2})^{95}(1/\sqrt{2} - i/\sqrt{2}) \\ &= 2^{47}(1 - i).\end{aligned}$$

6. **(a)** The polar form for $z_0 = -1$ is

$$-1 = e^{\pi i}.$$

Then by the identity $z^{1/m} = \sqrt[m]{|z|} e^{i(\theta + 2k\pi)/m}$ with $k = 0, 1, 2, \ldots, m-1$, we obtain

$$(-1)^{1/5} = e^{\pi i/5 + 2k\pi i/5},$$

where $k = 0, 1, 2, 3, 4$. Those are the fifth roots of $z_0 = -1$. Geometrically, we can locate each of the roots $e^{\pi i/5}$, $e^{\pi i/5 + 2\pi i/5}$, $e^{\pi i/5 + 2\cdot 2\pi i/5}$, $e^{\pi i/5 + 2\cdot 3\pi i/5}$, and $e^{\pi i/5 + 2\cdot 4\pi i/5}$, which are all on the unit circle, by looking at their arguments. They are the vertices of a regular pentagon inscribed in the unit circle, where the verticses are at $e^{\pi i/5}$, $e^{3\pi i/5}$, $e^{\pi i} = -1$, $e^{7\pi i/5}$, and $e^{9\pi i/5}$.

11. It follows from $(z+1)^5 = z^5$ that

$$\left(\frac{z+1}{z}\right)^5 = 1,$$

which means that $\dfrac{z+1}{z}$ may take the values of the fifth roots of unity $\omega = e^{2k\pi i/5}$, where $k = 0, 1, 2, 3, 4$. However, $\dfrac{z+1}{z}$ cannot take the value $e^{2\cdot 0\pi i/5} = 1$; Otherwise, $\dfrac{z+1}{z} = 1$ implies an absurd identity $1 = 0$. Therefore,

$$\frac{z+1}{z} = \omega$$

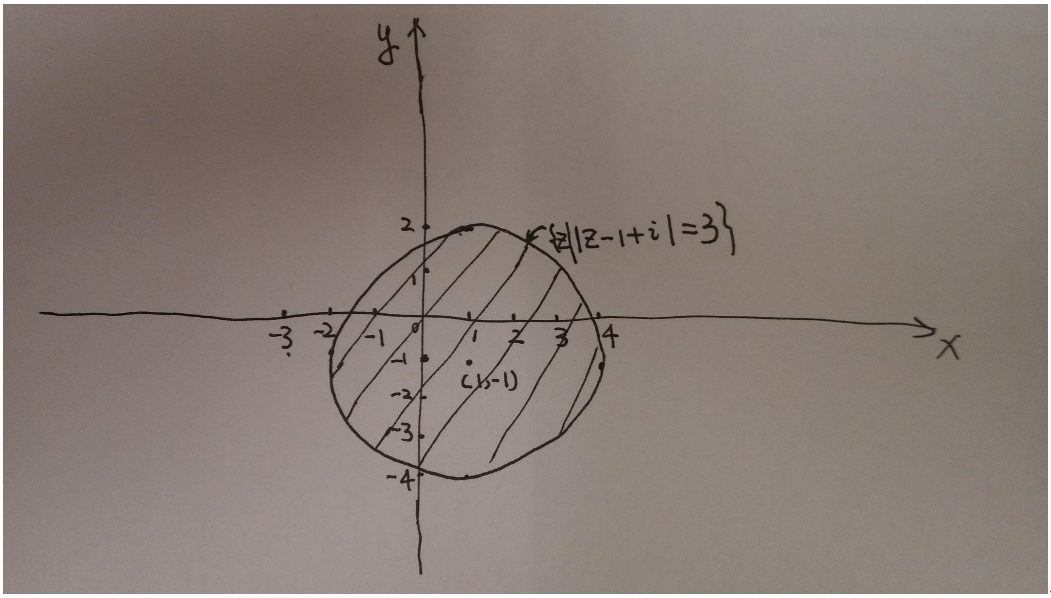
implies that

$$z = \frac{1}{\omega - 1},$$

where $\omega = e^{2k\pi i/5}$ with $k = 1, 2, 3, 4$.

EXERCISES 1.6

2. **(a)** Let $z = x + iy$. Then the sketch of the set $\{z : |z - 1 + i| \leqslant 3\}$ is something like



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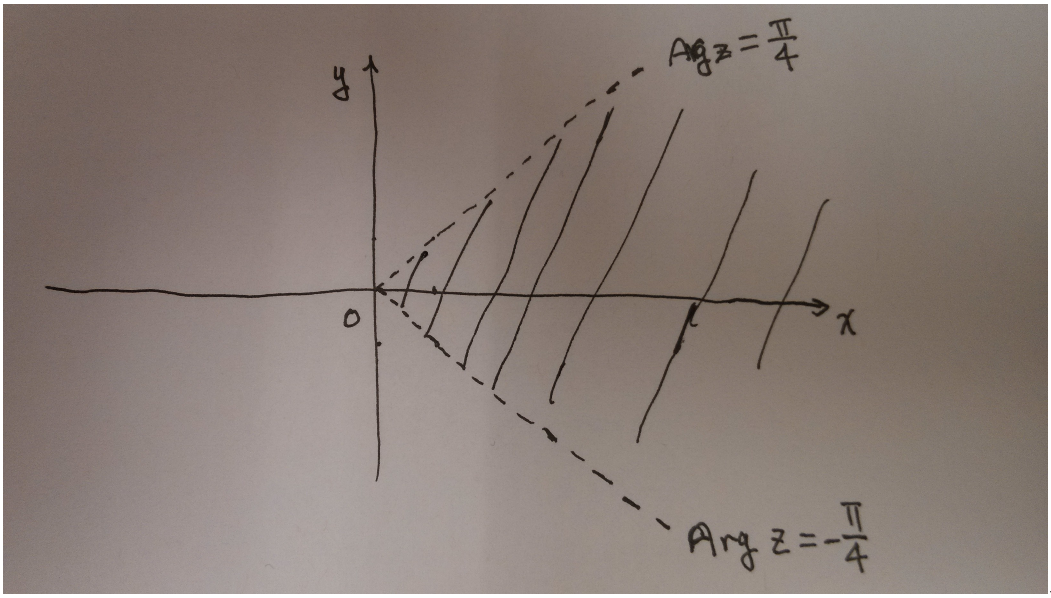
(b) Let $z = x + iy$. Then the sketch of the set $\{z : |\text{Arg } z| < \pi/4\}$ is something like



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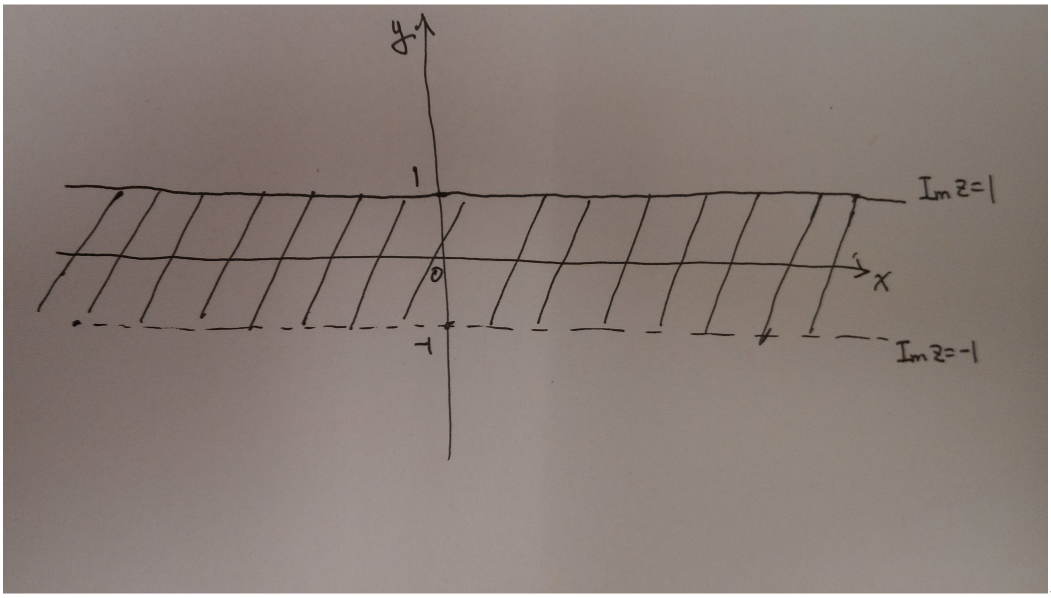
(d) Let $z = x + iy$. Then the sketch of the set $\{z : -1 < \text{Im } z \leqslant 1\}$ is something like



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3. Recall that for every point z_0 of an open set S, there is a circular neighbourhood of z, $B_\rho(z) = \{w : |w - z| < \rho\}$ with ρ being any small enough positive real number, such that $B_\rho(z)$ is completely contained in S. Then **(a)** is not open, since the points on the circle $\{z : |z-1+i| = 3\}$ do not have such neighbourhood. **(d)** is not open since the points on the line $\{z : \text{Im } z = 1\}$ do not have such neighbourhood. **(b)** satisfies this property. We conclude that **(b)** is open.

4. Recall that a domain is open connected. From the graphs above and Question **3**, we conclude that **(b)** is a domain, since for any two points z_1 and z_2, there is a polygonal path that lies entirely in $\{z : |\text{Arg } z| < \pi/4\}$ joining z_1 and z_2.

5. Recall that for every point z in a bounded set S, there is a positive real number R such that $|z| < R$. For **(a)**, we have $|z| \leqslant |1-i| + |z-1+i| \leqslant 2+3 = 5$. Hence R can be any real number greater than 5. But for **(b)** and **(d)** there is no such R. We conclude that **(a)** is bounded.

EXERCISE 2.1

1. **(a)** Let $z = x + iy$. Then

$$\begin{aligned} f(z) &= f(x+iy) = 3(x+iy)^2 + 5(x+iy) + i + 1 \\ &= (3x^2 - 3y^2 + 5x + 1) + i(6xy + 5y + 1). \end{aligned}$$

(c) Let $z = x + iy$. Then

$$\begin{aligned} h(z) &= \frac{z+i}{z^2+1} = \frac{1}{z-i} = \frac{1}{x + i(y-1)} \\ &= \frac{x - i(y-1)}{(x+i(y-1))(x-i(y-1))} \\ &= \frac{x}{x^2 + (y-1)^2} + i\frac{1-y}{x^2+(y-1)^2}. \end{aligned}$$

(f) Let $z = x + iy$. Then

$$\begin{aligned} G(z) &= G(x+iy) = e^x \cdot e^{iy} + e^{-x} \cdot e^{-iy} \\ &= e^x(\cos y + i \sin y) + e^{-x}(\cos(-y) + i \sin(-y)) \\ &= (e^x + e^{-x}) \cos y + i(e^x - e^{-x}) \sin y \\ &= 2 \cosh x \cos y + i2 \sinh x \sin y. \end{aligned}$$

2. **(a)** The domain of definition of $f(z)$ is the entire complex plain $\mathbb{C}$.

(c) The domain of definition of $h(z)$ is the entire complex plain $\mathbb{C}$ except for two points $z = i$ and $z = -i$, i.e., $\mathbb{C} \setminus \{i, -i\}$.

(f) The domain of definition of $G(z)$ is the entire complex plain $\mathbb{C}$.

4. Since the points on the circle $|z - 1| = 1$ can be written as the polar form

$$z = 1 + e^{i\theta},$$

where we take $0 \leqslant< \theta < 2\pi$, it follows that under the mapping $w = f(z) = 1/z$, those points are mapped to

$$\frac{1}{1 + e^{i\theta}}.$$

Then

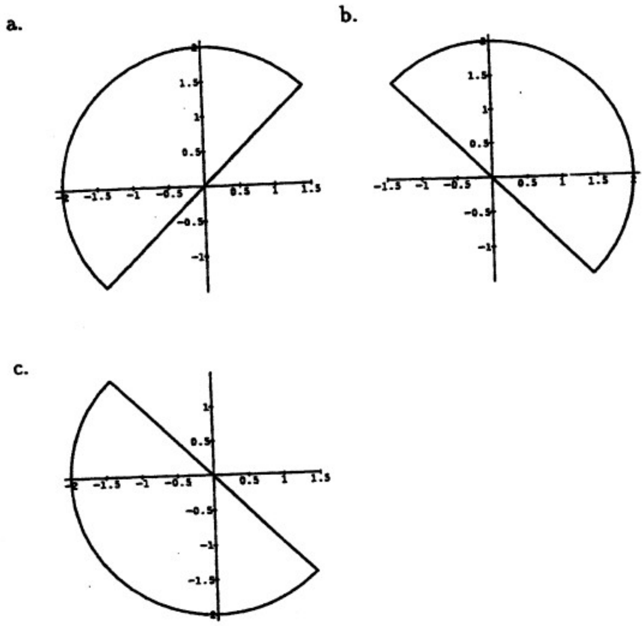
$$\begin{aligned} \frac{1}{1 + e^{i\theta}} &= \frac{1 + e^{-i\theta}}{(1 + e^{i\theta})(1 + e^{-i\theta})} \\ &= \frac{1 + \cos\theta - i\sin\theta}{2(1 + \cos\theta)} \\ &= \frac{1}{2} - i\frac{\sin\theta}{2(1 + \cos\theta)}. \end{aligned}$$

Hence we obtain that the image is a vertical line $x = 1/2$ since as $0 \leqslant \theta < 2\pi$, the range of $\dfrac{\sin\theta}{2(1 + \cos\theta)}$ is $\mathbb{R} \cup \{\infty\}$.
Or: $w = f(z) = \dfrac{1}{z}$ implies that

$$|z - 1| = |\frac{1}{w} - 1| = \frac{|w - 1|}{|w|} = 1.$$

Then $|w - 1| = |w|$, which indicates that the points w are equidistant from $(0, 0)$ and $(1, 0)$. Hence w is the line $x = \dfrac{1}{2}$.

8.



(a) is obtained by rotating the semidisk $\{|z| \leqslant 2, \mathrm{Im}\ z \geqslant 0\}$ $\pi/4$ counter-clockwise.

(b) is obtained by rotating the semidisk $\{|z| \leqslant 2, \mathrm{Im}\ z \geqslant 0\}$ $\pi/4$ clockwise.

(c) is obtained by rotating the semidisk $\{|z| \leqslant 2, \mathrm{Im}\ z \geqslant 0\}$ $3\pi/4$ counter-clockwise.