Homework 8 - Math 321, Spring 2012

Due on Friday March 16

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1. (a) Suppose f is a continuous function on $[1, \infty)$. For every real number $t \geq 1$, compute the Riemann-Stieltjes integral
$$F(t) = \int_1^t f(x)d[x],$$
where $[x]$ is the greatest integer in x.

   (b) Given a sequence $\{x_n : n \geq 1\}$ of distinct points in (a, b) and a sequence $\{c_n : n \geq 1\}$ of positive numbers with $\sum_{n=1}^{\infty} c_n < \infty$, define an increasing function α on $[a, b]$ by setting
$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad \text{where} \quad I(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$
Show that
$$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} c_n f(x_n)$$
for every continuous function f on $[a, b]$.

2. (a) If f and α share a common-sided discontinuity in $[a, b]$, show that f is not in $\mathcal{R}_\alpha[a, b]$.

   (b) Identify the class of functions that are Riemann-Stieltjes integrable on $[a, b]$ with respect to α for every nondecreasing α. In other words, describe the set
$$\bigcap \{\mathcal{R}_\alpha[a, b] : \alpha \text{ nondecreasing}\}.$$

   (c) Recall that $S[a, b]$ is the collection of all step functions on $[a, b]$. If
$$S[a, b] \subseteq \mathcal{R}_\alpha[a, b],$$
show that α is continuous.

3. We have seen $\chi_{\mathbb{Q}}$ (the indicator function of the rationals) is not Riemann integrable on $[0, 1]$. The problem was that it was too discontinuous - in fact, every point in $[0, 1]$ was a point of discontinuity. Here is another example of a function with uncountably many points of discontinuity, but this time Riemann integrable.

   Show that the set of discontinuities of χ_Δ (the indicator function of the Cantor middle-third set) is precisely Δ, which is an uncountable set, but that χ_Δ is nevertheless Riemann integrable on $[0, 1]$.