# Homework 7 - Math 321, Spring 2012

## Due on Friday March 9

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1. Show that BV$[a,b]$, the space of functions of bounded variation on $[a,b]$ is complete under the *variation norm* $||\cdot||_{\mathrm{BV}}$:
$$||f||_{\mathrm{BV}} = |f(a)| + V_a^b f.$$
2. Let us return to our discussion of Jordan's theorem. Convince yourself that the decomposition of a function of bounded variation into the difference of increasing functions is by no means unique: $f = g - h = (g+1) - (h+1)$. The purpose of this problem is to indicate a specific choice of $g$ and $h$ that turns out to be very useful in studying $BV[a,b]$.

   Given $f \in BV[a,b]$ and $v(x) = V_a^x f$, we define the *positive variation* of $f$ by
$$p(x) = \frac{1}{2}\left[v(x) + f(x) - f(a)\right]$$
   and the *negative variation* of $f$ by
$$n(x) = \frac{1}{2}\left[v(x) - f(x) + f(a)\right],$$
   so that $v(x) = p(x) + n(x)$ and $f(x) = f(a) + p(x) - n(x)$.

   (a) Prove that the functions $p$ and $n$ admit the following natural characterizations:
$$p(x) = \sup_{P\in\Pi^+(x)} \sum_j \left[f(d_j) - f(c_j)\right], \qquad n(x) = \sup_{P\in\Pi^-(x)} \sum_j \left[f(c_j) - f(d_j)\right],$$
   where
$$\Pi^+(x) = \Big\{P = \bigcup_{j=1}^n [c_j, d_j] \subseteq [a,x] : n \geq 1, a \leq c_j < d_j \leq c_{j+1} \leq b,$$
$$f(d_j) - f(c_j) \geq 0 \text{ for all } j\Big\},$$
$$\Pi^-(x) = \Big\{P = \bigcup_{j=1}^n [c_j, d_j] \subseteq [a,x] : n \geq 1, a \leq c_j < d_j \leq c_{j+1} \leq b,$$
$$f(d_j) - f(c_j) \leq 0 \text{ for all } j\Big\}.$$
   Use this description to deduce that $0 \leq p \leq v$ and $0 \leq n \leq v$ and that $p$ and $n$ are increasing functions on $[a,b]$.

   (b) Show that the functions $p$ and $n$ are "variation minimizers" for the Jordan decomposition of $f$, in the following sense: If $g$ and $h$ are increasing functions on $[a,b]$ such that $f = g-h$, then
$$V_x^y p \leq V_x^y g \quad \text{and} \quad V_x^y n \leq V_x^y h \quad \text{for all } x < y \text{ in } [a,b].$$
3. Prove the following result known as Helly's first theorem: Let $\{f_n\}$ be a bounded sequence in BV$[a,b]$, i.e., suppose that $||f_n||_{\mathrm{BV}} \leq K$ for all $n$. Then some subsequence of $\{f_n\}$ converges pointwise on $[a,b]$ to a function $f \in$ BV$[a,b]$, which also satisfies $||f||_{\mathrm{BV}} \leq K$. (Hint: Use Jordan's theorem to write $f_n = g_n - h_n$, where $g_n$ and $h_n$ are uniformly bounded sequences of increasing functions.)

Helly's first theorem should be viewed as an almost-compactness result in that it provides a convergent subsequence for any bounded sequence in $\mathrm{BV}[a, b]$. Unfortunately, the convergence here is pointwise and not necessarily convergence in the metric of $\mathrm{BV}[a, b]$, or even in the weaker metric of $\mathrm{B}[a, b]$.