Homework 6 - Math 321, Spring 2012

Due on Friday February 17

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1. Let $\mathcal{A}$ be a normed algebra. Show that if $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, then so is $\overline{\mathcal{B}}$.
2. Let $\mathcal{A}$ be a vector subspace of $B(X)$, the space of bounded real-valued functions on X. Show that $\mathcal{A}$ is a sublattice of $B(X)$ if and only if $\mathcal{A}$ is closed under absolute value; i.e., $|f| \in \mathcal{A}$ whenever $f \in \mathcal{A}$. If X is a compact metric space, use this to show that $C(X)$ is a sublattice of $B(X)$.
3. The version of Stone-Weierstrass theorem that we proved in class was for real scalars, i.e., the scalar field underlying the vector space $C(X)$ was $\mathbb{R}$. Use this "real version" to deduce the following "complex version" of Stone-Weierstrass theorem :

   Let X be a compact metric space, let $C_{\mathbb{C}}(X)$ denote the space of all complex-valued continuous functions on X, and let $A_{\mathbb{C}}$ be a subalgebra over $\mathbb{C}$ of $C_{\mathbb{C}}(X)$. If $A_{\mathbb{C}}$ separates points in X, vanishes at no point in X and is self-conjugate, then $A_{\mathbb{C}}$ is dense in $C_{\mathbb{C}}(X)$.
4. The space of polynomials in the complex variable $z = x + iy$ obviously separates points in $\mathbb{T} = \{e^{it} : 0 \leq t < 2\pi\}$ and vanishes at no point of $\mathbb{T}$ (convince yourself of this, but you need not submit a proof of it). Nevertheless, the polynomials in z (with complex coefficients) are not dense in the space of continuous complex-valued functions of $\mathbb{T}$. Here is a proof - fill in the steps outlined below to show that $f(z) = \overline{z}$ cannot be uniformly approximated by polynomials in z:
   (a) If $p(z) = \sum_{k=0}^{n} c_k z^k$, show that
   $$\int_0^{2\pi} \overline{f(e^{it})} p(e^{it})\, dt = 0.$$
   (b) Show that
   $$2\pi = \int_0^{2\pi} \overline{f(e^{it})} f(e^{it})\, dt = \int_0^{2\pi} \overline{f(e^{it})} \left[f(e^{it}) - p(e^{it}\right]\, dt.$$
   (c) Conclude that $||f - p||_\infty \geq 1$ for any polynomial p.
   (d) Why does the conclusion of part (c) not contradict the Stone-Weierstrass theorem?