Homework 3 - Math 321, Spring 2012

Due on Friday January 27

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1. Recall that
   - a metric space is separable if it has a countable dense subset, and
   - $B[0,1]$ denotes the space of bounded real-valued functions on $[0,1]$.

   Is $(B[0,1], ||\cdot||_\infty)$ separable? Give reasons for your answer.
2. Use convergence results proved in class to deduce the following properties of power series:
   (a) If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for some $x_0 \neq 0$, show that it converges uniformly and absolutely on every interval $[-r, r]$, where $0 < r < |x_0|$. Deduce that the sum represents a continuous function for $|x| < |x_0|$.
   (b) Show that term-by-term differentiation or integration works. Formulate mathematically what this statement means, with special attention to the domains where you carry out these operations, and then prove it.
3. Let $\mathcal{P}_n$ denote the space of polynomials of degree at most n, and let $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$. Answer the following questions, with justification:
   (a) Is $\mathcal{P}_n$ closed in $C[0,1]$?
   (b) Is $\mathcal{P}$ equal to, or a strict subset of $C[0,1]$?
4. Remember the Cantor $\frac{1}{3}$-set Δ? If not, review its definition on page 41 of the text.

   In class, we used the notion of uniform convergence to construct an example of a continuous, nowhere differentiable function on $\mathbb{R}$. Let us apply the same notion now towards another construction, namely that of a space-filling curve. Follow the steps outlined below to find a pair of continuous functions $x(t)$ and $y(t)$ on $[0,1]$ such that the curve $t \mapsto (x(t), y(t))$ fills the unit square $[0,1]\times[0,1]$; in fact the curve maps Δ onto $[0,1]\times[0,1]$.

   To begin with, define a map $f : \mathbb{R} \to [0,1]$ as follows. Let

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{3}, \\ 3t-1 & \text{for } \frac{1}{3} < t < \frac{2}{3}, \\ 1 & \text{for } \frac{2}{3} \le t \le 1. \end{cases}$$

   Extend f to all of $\mathbb{R}$ by taking f to be even and periodic of period 2.
   (a) Keeping in mind that any $t \in \Delta$ admits a ternary (in other words base 3) expansion

$$t = 0.(2a_0)(2a_1)(2a_2)\cdots \text{(base 3)}, \qquad \text{where each } a_k \text{ is either 0 or 1,}$$

   prove that $f(3^k t) = a_k$ for $t \in \Delta$ and all $k \ge 1$. This will be the basis of our construction.
   (b) Set

$$x(t) = \sum_{k=0}^{\infty} 2^{-k-1} f(3^{2k} t), \qquad y(t) = \sum_{k=0}^{\infty} 2^{-k-1} f(3^{2k+1} t)$$

   Show that x and y are continuous on all of $\mathbb{R}$, and maps $\mathbb{R}$ into $[0,1]$.
   (c) Use part (a) of this problem to show that given $x_0, y_0 \in [0,1]$, there exists $t_0 \in \Delta$ such that

$$x(t_0) = x_0, \qquad y(t_0) = y_0.$$

   Thus the curve maps Δ (and hence $[0,1]$) onto $[0,1]\times[0,1]$.