Problem Set 11 - Math 321, Spring 2012

This homework set is not meant to be turned in.
Use it as review for material covered during last week of classes.

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1. Let $\mathcal{R}[-\pi, \pi]$ denote the space of Riemann integrable functions on $[a, b]$. We introduced the notion of "L^2 norm" in class, namely,

$$||f||_2 = \left[\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx\right]^{\frac{1}{2}}, \qquad f \in \mathcal{R}[-\pi, \pi].$$

This problem is concerned with the modifications necessary to justify this nomenclature.
(a) If f is Riemann integrable on $[-\pi, \pi]$ and $||f||_2 = 0$, does it follow that $f \equiv 0$?
(b) If you assume in addition that f is continuous, show that the above implication is true. Use this to verify that the L^2 norm is truly a norm on $C[-\pi, \pi]$. Is $C[-\pi, \pi]$ closed under this norm?
(c) Define a binary relation $\sim$ on $\mathcal{R}[-\pi, \pi]$ as follows: for $f, g \in \mathcal{R}[-\pi, \pi]$, we say

$$f \sim g \quad \text{if} \quad \int_{-\pi}^{\pi} |f(x) - g(x)|^2\, dx = 0.$$

Show that $\sim$ is an equivalence relation, i.e, it is reflexive, symmetric and transitive.
(d) Given $f \in \mathcal{R}[-\pi, \pi]$, an equivalence class $\mathcal{F}$ of f is the set of all functions g such $f \sim g$. Define $L^2[-\pi, \pi]$ to be the space of equivalence classes of $\mathcal{R}[-\pi, \pi]$. Show that $L^2[-\pi, \pi]$ is a vector space, with the natural extensions of the notions of vector addition and scalar multiplication inherited from $\mathcal{R}[-\pi, \pi]$.
(e) For any $\mathcal{F} \in L^2[-\pi, \pi]$, define

$$||\mathcal{F}||_2 = \left[\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx\right]^{\frac{1}{2}},$$

where f is any member of the equivalence class $\mathcal{F}$. Show that $||\mathcal{F}||_2$ is well-defined, i.e., independent of the choice of $f \in \mathcal{F}$, and that $||\cdot||_2$ is a genuine norm on $L^2[-\pi, \pi]$.
(f) Show that $L^2[-\pi, \pi]$ equipped with $||\cdot||_2$ is a Banach space. (Remark: By a slight abuse of notation, one says $f \in L^2[-\pi, \pi]$ when one really means that the equivalence class of f is in $L^2[-\pi, pi]$.)

2. Our proof of the L^2-convergence of Fourier series relied on the following approximation result. Fill in the details.
   Let f be Riemann integrable on $[-\pi, \pi]$, and let $\epsilon > 0$.
(a) Show that there is a continuous function g on $[-\pi, \pi]$ satisfying $||f - g||_2 < \epsilon$.
(b) Show that there is a continuous, 2π-periodic $h \in \mathcal{C}^{2\pi}$ satisfying $||f - h||_2 < \epsilon$.
(c) Show that there is a trig polynomial T with $||f - T||_2 < \epsilon$.

3. If two Riemann integrable functions f and g share the same Fourier coefficients, does it follow that $f \equiv g$ as elements of $\mathcal{R}[-\pi, \pi]$? As elements of $L^2[-\pi, \pi]$? What would your answer be if f and g were required to be continuous?

4. In class, we dealt mainly with the issue of L^2-norm convergence of Fourier series. This problem addresses some aspects of Fourier series involving uniform convergence, obtainable as easy consequences of results we have learnt in this course.

(a) Show that if the Fourier series of a function $f \in \mathcal{C}^{2\pi}$ is uniformly convergent, then the series must actually converge to f.

(b) If the Fourier coefficients $\{a_n, b_n\}$ for some function $f \in \mathcal{C}^{2\pi}$ satisfy

$$\sum_n (|a_n| + |b_n|) < \infty,$$

show that the Fourier series for f converges uniformly to f.

(c) Define $f(x) = (\pi - x)^2$ for $0 \leq x \leq 2\pi$, and extend f to a 2π-periodic continuous function on $\mathbb{R}$ in the obvious way. Show that

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

Note that setting $x = 0$ yields the familiar formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

5. Let $\{a_n(f), b_n(f)\}$ denote the Fourier coefficients of f. Determine whether the mapping $f \mapsto \{a_n(f), b_n(f)\}$ is a surjective isometry of $L^2[-\pi, \pi]$ onto ℓ^2.
6. Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and Riemann integrable on $[-\pi, \pi]$. Prove that

$$\lim_{x \to 0} \int_{-\pi}^{\pi} |f(x + t) = f(t)|^2 \, dt = 0.$$