Midterm 1 Practice Problem set

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1. Let $\{P_n\}$ be a sequence of real polynomials of degree $\leq d$, a fixed integer. Suppose that $P_n(x) \to 0$ pointwise for $0 \leq x \leq 1$. Prove that $P_n \to 0$ uniformly on $[0,1]$.
2. For each positive integer n, define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \cos nx$. Prove that the sequence of functions $\{f_n\}$ has no uniformly convergent subsequence.
3. Let $C^1[a,b]$ be the space of real-valued continuous functions f on $[0,1]$ such that f' is continuous on $[0,1]$, with the norm
$$||f|| = \sup_{0\leq x\leq 1}\left|f(x)\right| + \sup_{0\leq x\leq 1}\left|f'(x)\right|.$$
Which subsets of $C^1[0,1]$ are compact?
4. Let the functions $f_n : [0,1] \to [0,1]$, $n = 1,2,3,\cdots$ satisfy
$$\left|f_n(x) - f_n(y)\right| \leq |x-y| \quad \text{whenever} \quad |x-y| \geq \frac{1}{n}.$$
Prove that the sequence has a uniformly convergent subsequence.
5. (a) Give an example of a sequence of C^1 functions $f_k : [0,\infty) \to \mathbb{R}$, $k = 0,1,2,\cdots$ such that $f_k(0) = 0$ for all k and $f_k'(x) \to f_0'(x)$ for all x as $k \to \infty$, but $f_k(x)$ does not converge to $f_0(x)$ for all x as $k \to \infty$.
   (b) State an extra condition which would imply that $f_k(x) \to f_0(x)$ for all x as $k \to \infty$.
6. Define two sequences $\{f_n\}$ and $\{g_n\}$ as follows:
$$f_n(x) = x\left(1 + \frac{1}{n}\right) \text{ for } x \in \mathbb{R},\ n = 1,2,\cdots$$
$$g_n(x) = \begin{cases} \frac{1}{n} & \text{if } x = 0 \text{ or } x \notin \mathbb{Q}, \\ b + \frac{1}{n} & \text{if } x = \frac{a}{b} \in \mathbb{Q},\ a \in \mathbb{Z}, b \in \mathbb{N},\ \gcd(a,b) = 1. \end{cases}$$
Let $h_n(x) = f_n(x)g_n(x)$. Show that f_n and g_n converge uniformly on every finite interval, but that h_n fails to converge uniformly on every finite interval.
7. (a) If X is infinite, show that $B(X)$ is not separable.
   (b) Is $C(\mathbb{R})$ separable?
8. Let $S[a,b]$ denote the space of all step functions on $[a,b]$. Show that $f \in \overline{S[a,b]}$ if and only if both these properties hold:
   i. $f(x+)$ and $f(x-)$ exist at each x in (a,b), and
   ii. $f(a+)$ and $f(b-)$ exist.