Question 1

a. (8 pts) Find constants A, B, C, and D so that

$$\frac{x^3 - 8x + 4}{x(x-1)(x-2)} = A + \frac{B}{x} + \frac{C}{x-1} + \frac{D}{x-2}.$$

Solution: Since the numerator and denominator both have degree 3, the first step is to perform a long division: the denominator expands to $x^3 - 3x^2 + 2x$, which divides once into the numerator $x^3 - 8x + 4$, leaving a remainder of $3x^2 - 10x + 4$. So far, we have

$$\frac{x^3 - 8x + 4}{x(x-1)(x-2)} = 1 + \frac{3x^2 - 10x + 4}{x(x-1)(x-2)},$$

and now the remaining fraction has a lower degree in the numerator than in the denominator, making it suitable for partial fraction decomposition. Thus $\boxed{A = 1}$, and we now look for B, C, and D so that

$$\frac{3x^2 - 10x + 4}{x(x-1)(x-2)} = \frac{B}{x} + \frac{C}{x-1} + \frac{D}{x-2}.$$

Multiply both sides by the common denominator $x(x-1)(x-2)$ to get

$$3x^2 - 10x + 4 = B(x-1)(x-2) + Cx(x-2) + Dx(x-1), \tag{1}$$

which must hold for all x. In particular, when $x = 0$, equation (1) gives $4 = B(-1)(-2) = 2B$, so $\boxed{B = 2}$. Similarly, when $x = 1$, equation (1) becomes $3 - 10 + 4 = C(1)(-1)$, which simplifies to $\boxed{C = 3}$. To find D, we can plug in $x = 2$ which gives $3(4) - 10(2) + 4 = D(2)(1)$, which gives $\boxed{D = -2}$.

To summarize, we have $A = 1$, $B = 2$, $C = 3$, and $D = -2$, that is:

$$\frac{x^3 - 8x + 4}{x(x-1)(x-2)} = 1 + \frac{2}{x} + \frac{3}{x-1} + \frac{-2}{x-2}.$$

b. (8 pts) Compute the indefinite integral

$$\int \sec^3 x \tan^3 x \, dx.$$

Solution: Since the exponent of $\tan x$ is odd, we should try the substitution $\boxed{u = \sec x}$ (since $du = \sec x \tan x \, dx$, this will make the exponent of $\tan x$ even, once we convert from du to dx). We have

$$\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \tan^2 x \, du = \int \sec^2 x (\sec^2 x - 1) \, du = \int u^2(u^2 - 1) \, du.$$

The substitution succeeded in making the integrand much simpler. The rest is very straightforward:

$$\int u^2(u^2-1) \, du = \int u^4 - u^2 \, du = \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C.$$

c. (8 pts) Suppose Simpson's Rule with $n = 20$ was used to estimate

$$\int_0^\pi x^4 + \sin(2x) \, dx.$$

Find, with justification, a reasonable upper bound for the absolute error of this approximation.

Solution: The absolute error in Simpson's Rule for $\int_a^b f(x) \, dx$ is guaranteed to be at most

$$\frac{K(b-a)}{180}(\Delta x)^4,$$

provided that $|f^{(4)}(x)| \le K$ for all x in the interval $[0, \pi]$. We know that $b = \pi$ and $a = 0$, so $\Delta x = (b-a)/n = \pi/20$. The main question is how to find a value for K. First we differentiate $f(x) = x^4 + \sin(2x)$:

$$f'(x) = 4x^3 + 2\cos(2x), \qquad f'''(x) = 24x - 8\cos(2x),$$
$$f''(x) = 12x^2 - 4\sin(2x), \qquad f^{(4)}(x) = 24 + 16\sin(2x).$$

The largest value that $\sin(2x)$ ever takes is ± 1. Therefore $|f^{(4)}(x)| = |24 + 16\sin(2x)|$ is never larger than $24 + 16 = 40$, and we can take $K = 40$. The worst-case error is therefore

$$\frac{40\pi}{180}(\pi/20)^4 = \frac{40\pi^5}{180 \cdot 20^4}.$$

[This can be simplified to $\pi^5/720000$, or about 0.000425 by calculator.]

d. (8 pts) Let Z be a continuous random variable with a standard normal distribution. Recall that the probability density function of Z is

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}.$$

Suppose we define a new function $H(x)$ in terms of the probability

$$H(x) = \Pr(-x^2 < Z < x^2).$$

Find a formula for $H'(x)$. Your final answer must not contain any derivative or integral signs.

Solution: The key here is that since $f(x)$ is the PDF of Z, we know that $\Pr(a < Z < b) = \int_a^b f(x)\,dx$ for any a and b, so in particular

$$H(x) = \Pr(-x^2 < Z < x^2) = \int_{-x^2}^{x^2} f(t)\,dt.$$

(We changed the integration variable to t to prevent confusing it with the x appearing the limits.) There's no convenient formula for the CDF $F(x)$, but we can still use the Fundamental Theorem of Calculus to write $H(x)$ in terms of $F(x)$:

$$H(x) = F(x^2) - F(-x^2).$$

By the chain rule,

$$H'(x) = F'(x^2)(2x) - F'(-x^2)(-2x) = 2xF'(x^2) + 2xF'(-x^2).$$

But we know that $F'(x) = f(x)$ (this is the definition of the PDF), so

$$\begin{aligned} H'(x) = 2xf(x^2) + 2xf(-x^2) &= 2x\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x^2)^2} + 2x\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(-x^2)^2} \\ &= \frac{4x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^4}. \end{aligned}$$

Since there are no integrals or derivatives in this expression, we're done.

e. (8 pts) Consider the improper integral

$$\int_{-1}^{8} x^{-5/3}\, dx.$$

Determine whether this integral converges or diverges. If it converges, then calculate its value.

Solution: This integral is improper because the integrand $x^{-5/3}$ is undefined at $x = 0$, and the interval of integration includes this point. To properly compute the integral, we split it into two parts

$$\int_{-1}^{0} x^{-5/3}\, dx + \int_{0}^{8} x^{-5/3}\, dx = \lim_{b\to 0^-} \int_{-1}^{b} x^{-5/3}\, dx + \lim_{a\to 0^+} \int_{a}^{8} x^{-5/3}\, dx.$$

Let's study the first of these two limits: the antiderivative of $x^{-5/3}$ is $-\frac{3}{2}x^{-2/3}$, so the first limit is equal to

$$\lim_{b\to 0^-} \int_{-1}^{b} x^{-5/3} = \lim_{b\to 0^-} \Big(-\tfrac{3}{2}x^{-2/3} \Big|_{x=-1}^{b} \Big) = \lim_{b\to 0^-} \Big(-\tfrac{3}{2}b^{-2/3} + \tfrac{3}{2} \Big).$$

As $b \to 0$ from the left, $b^{-2/3}$ gets larger and larger, so the limit does not exist (this is true regardless of which direction we approach $b = 0$). Therefore the first half of the integral diverges, and this is enough to make the entire improper integral divergent.

[If we had checked the second half of the integral, we would also have found that it diverges for much the same reason.]

f. (10 pts) Calculate the following integral using a suitable trigonometric substitution:

$$\int \frac{dx}{x^2\sqrt{x^2-1}}.$$

Simplify your final answer so that it doesn't contain any trigonometric functions.

Solution: The presence of x^2-1, especially inside a square root, strongly suggests the substitution $\boxed{x = \sec\theta}$, so that we can make use of the identity $\sec^2\theta - 1 = \tan^2\theta$. To make the substitution, we compute $dx = \sec\theta\tan\theta\,d\theta$, so that

$$\int \frac{dx}{x^2\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\tan\theta} = \int \frac{d\theta}{\sec\theta}.$$

To solve this new integral, it's easy to see that $1/\sec\theta = \cos\theta$, so

$$\int \frac{d\theta}{\sec\theta} = \int \cos\theta\,d\theta = \sin\theta + C = \sin(\sec^{-1}x) + C.$$

We're almost done, but we need to simplify $\sin(\sec^{-1}x)$ to eliminate the trig functions. The quickest way to do this is to label a right triangle with angle θ so that the hypotenuse is x and adjacent side is 1 (since $\sec\theta = \text{hyp}/\text{adj}$). This means the opposite side is $\sqrt{x^2-1}$, and we can then use $\sin\theta = \text{opp}/\text{hyp} = \frac{\sqrt{x^2-1}}{x}$.

Here's a different, algebraic approach to simplifying: since $x = \sec\theta = 1/\cos\theta$, we have $\cos\theta = 1/x$, and so $\sin\theta = \sqrt{1-(1/x)^2}$. Therefore

$$\sin(\sec^{-1}x) + C = \sqrt{1-(1/x)^2} + C = \frac{\sqrt{x^2-1}}{x} + C.$$

Question 2

(15 pts) Compute the definite integral

$$\int_0^{\pi/2} \left(\sin^3 x\right) e^{\cos x}\, dx.$$

Solution: We first make the substitution $\boxed{u = \cos x}$. This is the only way to reduce the complexity of the expression $e^{\cos x}$. To carry out the substitution, we compute $du = -\sin x\, dx$, and adjust the endpoints: $x = 0$ means $u = 1$, and $x = \pi/2$ means $u = 0$. So then

$$\begin{aligned}\int_0^{\pi/2} \left(\sin^3 x\right) e^{\cos x}\, dx &= \int_1^0 -\left(\sin^3 x\right) e^{\cos x} \frac{1}{\sin x}\, du \\ &= \int_1^0 -\left(\sin^2 x\right) e^{\cos x}\, du = \int_0^1 \left(\sin^2 x\right) e^{\cos x}\, du \\ &= \int_0^1 \left(1 - \cos^2 x\right) e^u\, du = \int_0^1 (1-u^2) e^u\, du.\end{aligned}$$

The substitution is done, but now we need to integrate by parts (we could expand the product first, but it won't make it any easier or harder). Instead of u and dv we'll call the two parts s and dt, since u is already taken:

$$\begin{aligned} s &= 1 - u^2 & dt &= e^u\, du \\ t &= e^u & ds &= -2u\, du. \end{aligned}$$

$$\int_0^1 (1-u^2)e^u\, du = (1-u^2)e^u \Big|_{u=0}^1 + \int_0^1 2ue^u\, du = -1 + 2\int_0^1 ue^u\, du. \quad (2)$$

To evaluate this new integral, we integrate by parts once more:

$$\begin{aligned} s &= u & dt &= e^u\, du \\ t &= e^u & ds &= du. \end{aligned}$$

$$\int_0^1 ue^u\, du = ue^u \Big|_{u=0}^1 - \int_0^1 e^u\, du = e - \left(e^u \Big|_{u=0}^1\right) = e - (e-1) = 1.$$

Plugging this back into (2) gives

$$\int_0^1 (1-u^2)e^u\, du = -1 + 2(1) = 1,$$

and this is equal to the integral we started with. So the final answer is 1.

Question 3

A certain continuous random variable X is known to have a cumulative distribution function of the form

$$F(x) = \begin{cases} a, & \text{if } x < 0; \\ \frac{1}{2}x^2 + kx, & \text{if } 0 \le x \le 1; \\ b, & \text{if } x > 1, \end{cases}$$

where a, b, and k are constants.

a. (6 pts) Determine the exact values of a, b, and k using the fact that $F(x)$ is the CDF of the continuous random variable X.

Solution: Since $\lim_{x\to-\infty} F(x) = 0$ for any CDF and for this F we have $\lim_{x\to-\infty} F(x) = a$, we must have $\boxed{a = 0}$. Similarly, $\lim_{x\to-\infty} F(x) = b$, so $\boxed{b = 1}$. Finally, we need $F(x)$ to be continuous: so $\lim_{x\to 1^-} F(x) = \frac{1}{2} + k$ must agree with $\lim_{x\to 1^+} F(x) = b = 1$. Therefore $\boxed{k = \frac{1}{2}}$.

b. (6 pts) What is the expected value of X?

Solution: First we calculate the PDF of X by differentiating $F(x)$:

$$f(x) = \begin{cases} 0, & \text{if } x < 0; \\ x + \frac{1}{2}, & \text{if } 0 < x < 1; \\ 0, & \text{if } x > 1. \end{cases}$$

With PDF in hand, we can now easily calculate $\mathbb{E}(X)$ by the standard formula (note that $f(x)$ is exactly 0 unless x is between 0 and 1):

$$\begin{aligned} \mathbb{E}(X) &= \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x f(x)\,dx = \int_0^1 x(x + \tfrac{1}{2})\,dx \\ &= \int_0^1 x^2 + \tfrac{1}{2}x\,dx \;=\; \tfrac{1}{3}x^3 + \tfrac{1}{4}x^2 \Big|_{x=0}^{1} \;=\; \frac{1}{3} + \frac{1}{4} - 0 = \frac{7}{12}. \end{aligned}$$

c. (8 pts) What is the the standard deviation of X?

Solution: To calculate $\sigma(X)$, it is always easier to first calculate the variance $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$. We already found $\mathbb{E}(X)$ in part (b), so let's start by finding

$$\begin{aligned}\mathbb{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2(x+\tfrac{1}{2})\,dx \\ &= \int_0^1 x^3 + \tfrac{1}{2}x^2\,dx \;=\; \tfrac{1}{4}x^4 + \tfrac{1}{6}x^3 \Big|_{x=0}^{1} \;=\; \frac{1}{4} + \frac{1}{6} - 0 = \frac{5}{12}.\end{aligned}$$

So that gives

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{5\cdot 12 - 7^2}{12\cdot 12} = \frac{11}{12\cdot 12}.$$

Finally, we get the standard deviation of X by taking the square root: $\boxed{\sigma(x) = \sqrt{\mathrm{Var}(X)} = \sqrt{11}/12}$.

Question 4

(15 pts) Find the solution of the differential equation

$$\frac{dy}{dt} = e^{y-\ln t}$$

that satisfies the initial condition $y(1) = -1$.

Solution: We first look for the general solution to the differential equation $\frac{dy}{dt} = e^{y-\ln t}$. To separate y and t, we rewrite $e^{y-\ln t}$ as a product, namely

$$\frac{dy}{dt} = e^{y-\ln t} = e^{y}e^{-\ln t} = \frac{e^{y}}{t}.$$

Now it's easy to separate the y's and t's, by multiplying both sides by $e^{-y}\,dt$:

$$e^{-y}dy = \frac{dt}{t}.$$

After taking integrals, this becomes

$$-e^{-y} = \int e^{-y}dy = \int \frac{dt}{t}\,dt = \ln|t| + C.$$

At this point we could identify the constant C using the initial condition: when $t = 1$ we must have $y = -1$, so

$$-e^{1} = \ln|1| + C = C \implies C = -e.$$

We now finish the job by isolating for y:

$$\begin{aligned} e^{-y} &= -\ln|t| - C = e - \ln|t|, \\ -y &= \ln(e - \ln|t|), \\ y &= -\ln(e - \ln|t|). \end{aligned}$$

So the solution is given by $\boxed{y(t) = -\ln(e - \ln|t|)}$.

Warning: Don't be fooled into thinking this is the same as $-\ln e + \ln\ln|t|$; that would be $-\ln(e/\ln|t|)$, which is quite different from $-\ln(e - \ln|t|)$!