

**THE UNIVERSITY OF BRITISH COLUMBIA**

Sample Questions for Midterm 1 - January 26, 2012

**MATH 105 All Sections**

Closed book examination

Time: 50 minutes

**Last Name** \_\_\_\_\_ **First** \_\_\_\_\_

**Signature** \_\_\_\_\_

**Student Number** \_\_\_\_\_

**Special Instructions:**

No memory aids are allowed. One basic scientific calculator, WITH COVER REMOVED, may be used. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. Where boxes are provided for answers, put your final answers in them.

**Rules governing examinations**

- Each candidate must be prepared to produce, upon request, a UBCCard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  - (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  - (b) Speaking or communicating with other candidates.
  - (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1. Let  $P = (1, 0, 0)$  be a point and  $\bar{n} = \langle 2, 2, 1 \rangle$  a vector.

(a) Describe an equation of a plane passing through  $P$  with normal vector  $\bar{n}$ .

(b) Does the equation  $-6x - 6y - 3z = -6$  describes the same plane? Justify your answer.

(c) Compute a unit vector that shows in the same direction as  $\bar{n}$ .

**Solution:**

(a) One possible solution could be

$$2x + 2y + z = 2$$

Note, that the coefficients preceding  $x$ ,  $y$  and  $z$  are the components of the vector  $\bar{n}$  and the right-hand side of the equation is the dot product of  $\bar{n}$  with the vector defined by  $P$ , i. e.,  $\langle 2, 2, 1 \rangle \bullet \langle 1, 0, 0 \rangle$ .

(b) Yes, the equations describe the same plane because the normal vector given by the equation in (b) is  $\bar{m} = \langle -6, -6, -3 \rangle$  satisfies

$$\bar{m} = -3 \cdot \bar{n}$$

and the point  $P$  in (a) is contained in the plane given by the equation in (b).

(c) We compute the length of  $\bar{n}$  with the formula

$$|\bar{n}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

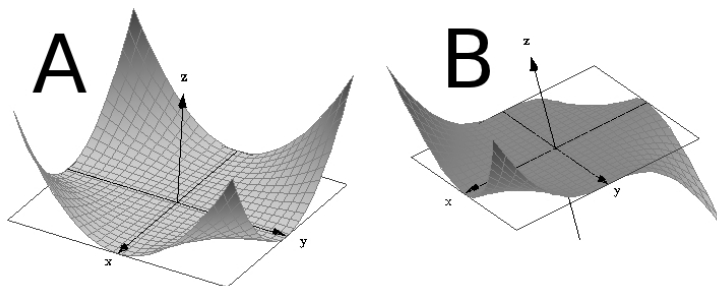
Hence, a unit vector showing in the same direction as  $\bar{n}$  is given by the vector

$$\frac{1}{3} \cdot \bar{n}$$

2. Let  $f(x, y) = x^2y^2$  be a function.

(a) Sketch a diagram with at least 3 level curves of  $f$ .

(b) Which of the following two surfaces correspond to the graph of  $f$ ? Decide and justify your answer!



**Solutions:**

(a) Level curves are the solutions (i. e., points  $(x, y)$ ) of  $c = f(x, y)$  for some fixed  $c$ . For  $c = 0$  we must solve  $0 = x^2y^2$ . The right-hand side is zero whenever  $x$  or  $y$  is zero. Hence, the points  $(x, 0)$  and  $(0, y)$  for arbitrary  $x$  and  $y$  are solutions to  $0 = f(x, y)$ . These points are exactly the  $x$ -axis ( $y = 0$ ) and  $y$ -axis ( $x = 0$ ). Let us pick  $c = 1$ . Then we have to solve for  $1 = x^2y^2$  and we get

$$y^2 = \frac{1}{x^2}$$

and taking the square root results in two solutions

$$y = \frac{1}{x} \quad \text{and} \quad y = -\frac{1}{x}$$

Setting  $c = 2$  and using the same steps above results in

$$y = \frac{\sqrt{2}}{x} \quad \text{and} \quad y = -\frac{\sqrt{2}}{x}$$

- (b) To decide which of the given surfaces is actually the graph of  $f(x, y)$  we compute the trace of the graph of  $f$  given by  $y = 1$ . We choose the plane because comparing the two pictures we see that fixing  $y$  yields two different traces. Note, that a plane given by  $y = 1$  is a plane parallel to the  $xz$ -plane with fixed  $y$ -value 1. We get  $z = f(x, 1) = x^2$ . Since this trace takes always non-negative values, the graph of  $f$  must be picture *A* because in picture *B* the trace for  $y = 1$  attains negative values as well.

3. Let  $f(x, y) = e^{\ln(x^2 - y^2)}$ .

(a) In the  $xy$ -plane sketch the domain of the function  $f$ .

(b) Show or disprove that  $f$  is a differentiable function.

(c) Show or disprove that there exist a differentiable function  $f$  satisfying

$$f_x = y^{2013} \quad \text{and} \quad f_y = 2012xy^{2012}.$$

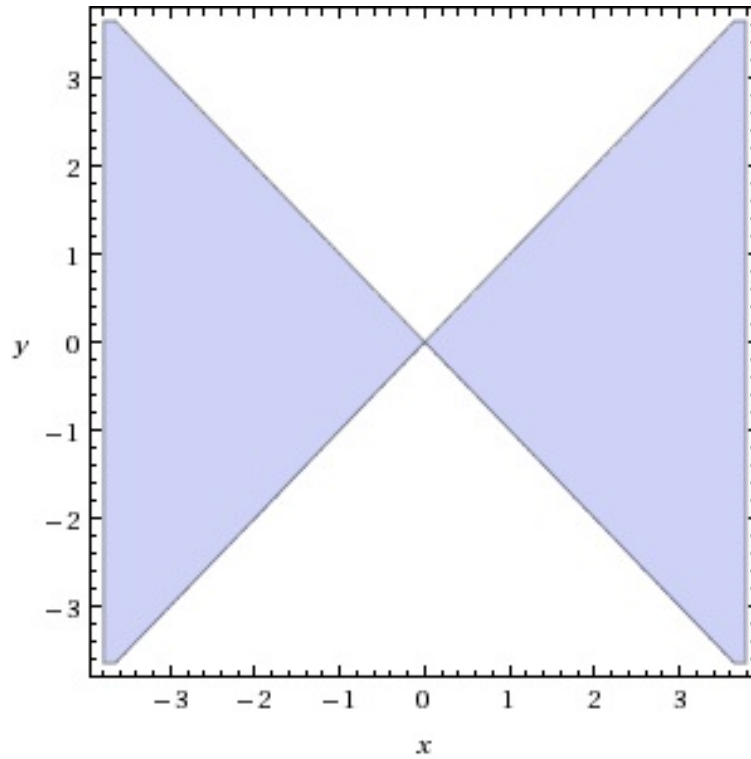
### Solutions:

(a) We must have  $x^2 - y^2 > 0$  because  $\ln$  has domain all positive real numbers. The points  $(x, y)$  satisfying this inequality form the blue shaded area in the following picture:

(b) One way of showing that  $f$  is a differentiable function is to compute the partial derivatives and show continuity of them because this implies that  $f$  is differentiable. The partial derivatives are:

$$f_x = \partial_x f = \frac{2xe^{\ln(x^2 - y^2)}}{x^2 - y^2}$$
$$f_y = \partial_y f = -\frac{2ye^{\ln(x^2 - y^2)}}{x^2 - y^2}$$

For all points in the domain numerator and denominator are continuous functions therefore  $f_x$  and  $f_y$  are. Another possibility to write down the partial derivatives of  $f$  is by recognising that  $e^{\ln(x^2 - y^2)} = x^2 - y^2$  and hence  $f_x = 2x$  and  $f_y = -2y$ .



(c) There is no differentiable function with this property because we must have  $f_{xy} = f_{yx}$ .  
In our example we have

$$f_{xy} = 2013y^{2012}$$

but

$$f_{yx} = 2012y^{2012}$$

and these two functions are not the same.

4. Let

$$f(x, y) = 4 + x^3 + y^3 - 3xy$$

be a function in two variables. Classify all critical points! **Solutions:** We first find all critical

points and apply afterwards the second derivative test to our results. We must compute all first and second order partial derivatives.

$$f_x = 3x^2 - 3y$$

$$f_y = 3y^2 - 3x$$

$$f_{xy} = -3$$

$$f_{xy} = -3$$

$$f_{xx} = 6x$$

$$f_{yy} = 6y$$

Critical points are points  $(x, y)$  that satisfy  $f_x(x, y) = 0 = f_y(x, y)$ . From the first equation  $f_x = 3x^2 - 3y = 0$  we get  $y = x^2$ . By replacing  $y$  with the right-hand side of this equation in the second equation of above we get  $f_y = 3y^2 - 3x = 3x^4 - 3x = 0$  or written as a product  $3x(x^3 - 1) = 0$ . There are only two solutions  $x = 0$  or solutions to  $x^3 - 1 = 0$ . The latter equation gives us  $x = 1$ . We get  $y = 0$  or  $y = 1$  respectively. Thus, the critical points of  $f$  are given by  $(0, 0)$  and  $(1, 1)$ .

We know must compute the discriminant  $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$  to test for local extrema or saddle points. We have

$$D(x, y) = 36x^2 \cdot 36y^2 - 9$$

Plugging-in the point  $(0, 0)$  yields  $D(0, 0) = -9 < 0$ . Hence,  $(0, 0)$  must be a saddle point. Plugging-in  $(1, 1)$  yields  $D(1, 1) = 36^2 - 9 > 0$ . Since  $f_{xx}(1, 1) > 0$  the point  $(1, 1)$  is a local minimum of  $f$  with minimum value  $f(1, 1) = 3$ .

5. Find the point on the line passing through  $(0, 0)$  with slope 3 that is closest to the point  $P = (1, 1)$ . **Solutions:** The question asks us to minimise the distance of a point on the line

described by  $y = 3x$  (this is the result of passing through the  $(0, 0)$  with slope 3) to the fixed point  $P = (1, 1)$ . The "distance-to- $P$ "-function in the plane to an arbitrary point  $(x, y)$  is given by the formula

$$d(x, y) = \sqrt{(x-1)^2 + (y-1)^2}$$

We do not want to minimise  $d$  for all points  $(x, y)$  just for the points lying on the line described by  $y = 3x$ . Hence, the last equation is a constraint and we define the constraint function  $g(x, y) = 3x - y$ . Points satisfying  $g(x, y) = 0$  satisfy  $3x - y$ . That explains the choice of  $g$  (We always want to have a constraint functions such that the constraint equation becomes  $g(x, y) = 0$ ). We use Lagrange-Multiplier to compute the minimum of  $d(x, y)$  restricted to the line given by  $g(x, y) = 0$ . The partial derivatives are:

$$\begin{aligned} d_x &= \frac{2(x-1)}{2 \cdot ((x-1)^2 + (y-1)^2)} = \frac{x-1}{(x-1)^2 + (y-1)^2} \\ d_y &= \frac{2(y-1)}{2 \cdot ((x-1)^2 + (y-1)^2)} = \frac{y-1}{(x-1)^2 + (y-1)^2} \\ g_x &= 3 \\ g_y &= -1 \end{aligned}$$

The method of Lagrange multiplier gives us the following 3 equations:

$$\begin{aligned} d_x &= \lambda \cdot g_x & \frac{x-1}{(x-1)^2 + (y-1)^2} &= 3\lambda \\ d_y &= \lambda \cdot g_y & \frac{y-1}{(x-1)^2 + (y-1)^2} &= -\lambda \\ g(x, y) &= 0 & 3x - y &= 0 \end{aligned}$$

We have to solve for  $x$  and  $y$ . With the first equation we can write  $\lambda$  as

$$\lambda = \frac{x-1}{3((x-1)^2 + (y-1)^2)}$$

We use this and replace  $\lambda$  in the second equation by the term on the right-hand side. and get

$$\frac{y-1}{(x-1)^2 + (y-1)^2} = -\frac{x-1}{3((x-1)^2 + (y-1)^2)}$$

If we multiply away the denominators we arrive at

$$y-1 = -\frac{x-1}{3}$$

or equivalently at

$$y = -\frac{x-1}{3} + 1 = \frac{4-x}{3} (*)$$



Since (always) the third equation is only a relation in  $x$  and  $y$  without  $\lambda$  we can replace  $y$  with the term on the right-hand side in the third equation.

$$3x - \frac{4-x}{3} = 0$$

The solution to this equation is seen to be  $x = 2/5$ . Now we plug this into (\*) and get  $y = 6/5$ . This is our solution. We can argue that this is a minimum because it is the only absolute extremum we have and geometrically every other point around  $(2/5, 6/5)$  must have larger distance to  $P$ .

6. Let  $[1, 2]$  be an interval with regular partition in 4 subintervals of equal length  $\Delta x = \frac{1}{4}$ . Find the function whose left Riemann Sum for this regular partition is given by

$$R = \frac{1}{4} \cdot \frac{1}{1} + \frac{1}{4} \cdot \frac{1}{1 + \frac{1}{4}} + \frac{1}{4} \cdot \frac{1}{1 + \frac{2}{4}} + \frac{1}{4} \cdot \frac{1}{1 + \frac{3}{4}} + \frac{1}{4} \cdot \frac{1}{2}$$

**Solutions:** The terms in a Riemann sum associated to a regular partition into 4 parts of

the interval  $[1, 2]$  always look like

$$f(\bar{x}_k) \cdot \Delta x = f(\bar{x}_k) \cdot \frac{1}{4}$$

because  $\Delta x = 1/4$  in this example. The grid points of the partitions are  $x_0 = 1$ ,  $x_1 = 5/4$ ,  $x_2 = 3/2$ ,  $x_3 = 7/4$  and  $x_4 = 2$ . Since we have given the left Riemann sum, we must have  $\bar{x}_1 = x_0$ ,  $\bar{x}_2 = x_1$ ,  $\bar{x}_3 = x_2$  and  $\bar{x}_4 = x_3$ . Hence

$$\begin{aligned} f(\bar{x}_1) &= f(x_0) && = f(1) = 1 \\ f(\bar{x}_2) &= f(x_1) && = f(5/4) = \frac{1}{1 + \frac{1}{4}} = \frac{1}{\frac{5}{4}} \\ f(\bar{x}_3) &= f(x_2) && = f(3/2) = \frac{1}{1 + \frac{2}{4}} = \frac{1}{\frac{3}{2}} \\ f(\bar{x}_4) &= f(x_3) && = f(7/4) = \frac{1}{1 + \frac{3}{4}} = \frac{1}{\frac{7}{4}} \end{aligned}$$

By comparing the things we plug-into  $f$  and our results, we see that  $f(x) = 1/x$  describes the left Riemann sum above.

7. Let  $f(x)$  be a differentiable function defined on some interval  $[a, b]$ . Assume that  $f'(x) > 0$  for all  $x$  in  $[a, b]$ . What can you say about the left-, right- and midpoint Riemann Sum?

**Solutions:** Choose some regular partition into  $n$  subintervals. Since the  $f$  is increasing

( $f'(x) > 0$ ) the values of  $f$  at the left endpoints of the subintervals are less than every other value of  $f$  on the subinterval. Hence, the areas of the rectangles (or the terms in the Riemann sum) must be smaller than the area under the graph of  $f$  on these subintervals. Thus, the left Riemann sum is an underestimation in this case.

By a similar argument (the values of  $f$  at the right endpoints of the subintervals are always larger than the values of  $f$  on the subintervals) we have that the right Riemann sum is an overestimate.

In general we do not know if the midpoint Riemann sum is an over- or underestimate.

**The End**