Name: ______________________________

MATH 105: Midterm #1 Practice Problems

1. `TRUE` or `FALSE`, **plus explanation**. Give a full-word answer `TRUE` or `FALSE`. If the statement is true, explain why, using concepts and results from class to justify your answer. If the statement is false, give a counterexample.

   (a) [4 points] Suppose the graph of a function f has the following properties: The trace in the plane $z = c$ is empty when $c < 0$, is a single point when $c = 0$, and is a circle when $c > 0$. Then the graph of f is a cone that opens upward.

   (b) [4 points] If $f(x, y)$ is any function of two variables, then no two level curves of f can intersect.

   (c) [4 points] Suppose P_1, P_2, and P_3 are three planes in $\mathbf{R}^3$. If P_1 and P_2 are both orthogonal to P_3, then P_1 and P_2 are parallel to each other.

(d) [4 points] If $f(x, y)$ has continuous partial derivatives of all orders, then $f_{xxy} = f_{yxx}$ at every point in $\mathbf{R}^2$.

(e) [4 points] Suppose that f is defined and differentiable on all of $\mathbf{R}^2$. If there are no critical points of f, then f does not have a global maximum on $\mathbf{R}^2$.

2. [5 points] Consider the function $f(x, y) = e^{y-x^2-1}$. Find the equation of the level curve of f that passes through the point $(2, 5)$. Then sketch this curve, clearly labeling the point $(2, 5)$.

3. Let $f(x, y) = (1 - 2y)(x^2 - xy)$.

   (a) [5 points] Compute the partial derivatives f_x and f_y.

   (b) [5 points] Using your answer to (a), find all the critical points of f.

(c) [5 points] Apply the second derivative test to label each of the points found in (b) as a `local minimum`, `local maximum`, `saddle point`, or `inconclusive`.

4. [15 points] Find the point (x, y, z) on the plane $x - 2y + 2z = 3$ that is closest to the origin. Show your work and explain your steps.

5. (a) [5 points] In your own words, explain what it means for a function $f(x, y)$ to have a `saddle point` at (a, b).

(b) [5 points] The function $f(x, y) = x^5 - x^2y^3 + y^7 + 11$ has a critical point at $(0, 0)$. (You may assume this without checking it.) Show that $(0, 0)$ is a saddle point of f.

6. [10 points] Find the absolute maximum value of the function $f(x, y) = xy^2$ on the region R consisting of those points (x, y) with $x^2 + y^2 \leq 4$ and $x \geq 0, y \geq 0$. (So R is the portion of the disk of radius 2 centered at the origin which belongs to the first quadrant, boundary points included.) Show your work and explain which methods from class you use.

7. [10 points] A firm makes x units of product A and y units of product B and has a production possibilities curve given by the equation $x^2 + 25y^2 = 25000$ for $x \geq 0, y \geq 0$. Suppose profits are \$3 per unit for product A and \$5 per unit for product B. Find the production schedule (i.e. the values of x and y) that maximizes the total profit.

8. In this problem, we guide you through the computation of the area underneath one hump of the curve $y = \sin x$.

   (a) [5 points] Write down the `right-endpoint Riemann sum` for the area under the graph of $y = \sin x$ from $x = 0$ to $x = \pi$, using n subintervals.

   (b) [10 points] Now assume the validity of the following formula (for each real number θ):

$$\sum_{k=1}^{n} \sin(k\theta) = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin(\frac{1}{2}(n+1)\theta).$$

   Using this formula, compute the limit as $n \to \infty$ of the expression found in part (a) and thus evaluate the area exactly. [*Hint:* The identity $\sin(\frac{\pi}{2} + \frac{\pi}{2n}) = \cos(\frac{\pi}{2n})$ may be useful.]