Final Exam Review Sheet - Math 105

Name: **SID:**

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1. The output of a manufacturing system is given by the production function

$$P = 10K^{\frac{1}{3}}L^{\frac{2}{3}},$$

where K represents the amount of invested capital and L represents the amount of labor. If every unit of capital and every unit of labor costs \$30 million and \$60 million respectively, find the maximum amount of production a company can achieve if it has at most \$360 million at its disposal.

(Answer: 40 units)

2. A function F in two variables is defined as follows,

$$F(x,y) = x^2 y \int_0^{\cos y} \ln(3+t^2)\,dt.$$

Find the partial derivatives $F_x(1, \frac{\pi}{2})$ and $F_y(1, \frac{\pi}{2})$.

(Answer: $F_x(1, \frac{\pi}{2}) = 0$, $F_y(1, \frac{\pi}{2}) = -\frac{\pi}{2}\ln 3$)

3. (a) A plane passing through the point $Q(1,0,1)$ is orthogonal to the vector joining the two points $Q_1(1,-1,2)$ and $Q_2(2,1,3)$. Find the equation of the plane.

(Answer: $x + 2y + z = 2$)

(b) For the plane whose equation you derived in part (a), find the point(s) on the plane closest to the point $P(2,0,4)$.

(Answer: $(4/3, -4/3, 10/3)$)

4. Evaluate the following integrals.

(a)

$$\int \frac{e^{2t}}{(1+e^{4t})^{\frac{3}{2}}}\,dt.$$

(Answer: $\frac{e^{2t}}{2\sqrt{1+e^{4t}}} + C$)

(b)

$$\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta.$$

(Answer: $\frac{\pi}{64}$)

(c)

$$\int \frac{3x^2+5x-2}{x^2-1}\,dx.$$

(Answer: $3x + 3\ln|x-1| + 2\ln|x+1| + C$)

5. Show whether the following improper integral

$$\int_0^{\infty} e^{-x}(\sin x + \cos x)\,dx$$

converges or diverges. If it converges, find its value.

(Answer: converges, value = 1/2)

6. Solve the initial value problem

$$\frac{dy}{dt} = \frac{t+1}{2ty}, \qquad y(1) = 4.$$

(Answer: $y = \sqrt{t + \ln t + 15}$)

7. Find a function f such that for every interval $[a, b]$ and for every n, the Simpson's rule approximation S_n gives the exact value of the integral $\int_a^b f(x)\, dx$.

(Answer: any polynomial of degree ≤ 3)

8. Find the limit

$$\lim_{h \to 0} \frac{1}{h} \int_0^h \arctan(t^2 + 1)\, dt.$$

(Answer: $\frac{\pi}{4}$)

9. (a) A discrete random variable takes only two values a and b with equal probabilities. The expected value of the random variable is $1/2$ and its standard deviation is $1/2$. Find a and b.

(Answer: 0 and 1)

(b) None of the functions below can be pdf-s of a continuous random variable for any choice of k. Explain why.

$$f(x) = \begin{cases} kx & \text{if } -2 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(Answer: f takes negative values)

$$g(x) = \begin{cases} \frac{2x}{k^2 - x^2} & \text{if } 0 \leq x \leq \frac{k}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

(Answer: $\int_0^{k/2} g(x)\, dx \neq 1$)

(c) A continuous random variable X has the property that

$$P(X > \ln x) = \frac{1}{x} \qquad \text{for } x > 1.$$

Find the pdf of X and use this to compute the expected value of X.

(Answer: $f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$, $\mathbb{E}(X) = 1$)

10. (a) Manipulate the power series expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \qquad |x| < 1$$

to derive the following Maclaurin series of the functions $g(x) = x/(1 + x^2)$ and $h(x) = \ln(1 + x^2)$:

$$g(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k+1},$$

$$h(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k}}{k}.$$

(b) Use the results in part (a) to evaluate the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{25^k}{k}$$

(Answer: $\ln(1/26)$)

11. Determine, with justification, whether each of the following series converges or diverges.

$$\text{(a)} \sum_{k=1}^{\infty} k\sin(\frac{1}{k}), \qquad \text{(b)} \sum_{k=1}^{\infty} k^2 e^{-k}, \qquad \text{(c)} \sum_{k=1}^{\infty} \frac{k\sin(k)}{k^3+1}, \qquad \text{(d)} \sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right).$$

(Answer: (a) diverges, (b) converges, (c) converges, (d) diverges)

12. Determine, with justification, whether each of the following sequences converges or diverges. If the sequence converges, find the limit.

$$\text{(a) } a_k = (1+2(-1)^k)\frac{k}{k^2+1},$$

$$\text{(b) } a_k = \sin\left(\frac{k\pi}{2}\right)\frac{k^2}{k^2+1},$$

$$\text{(c) } a_k = \frac{\sqrt{3k+1}-\sqrt{3k}}{\sqrt{4k+2}-2\sqrt{k}}.$$

(Answer: (a) converges to 0, (b) diverges, (c) converges to $\frac{1}{\sqrt{3}}$)