Math 105 Assignment 2 Solutions

Consider the function $f : [0, 2] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} \sqrt{1-(x-1)^2}, & 0 \le x \le 1, \\ x, & 1 < x \le 2. \end{cases}$$

(a) Sketch the graph of the function f (2 pts).

(b) Find an antiderivative of $\sqrt{1-(x-1)^2}$ (2 pts). **Hint**: use the formula

$$\int \sqrt{1-x^2}dx = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2} + C.$$

Solution: We want to do a substitution to get this in the form of the hint. Let $u = x - 1$, so $du = dx$, and appyling the substitution rule,

$$\int \sqrt{1-(x-1)^2}dx = \int \sqrt{1-u^2}du = \frac{u\sqrt{1-u^2}}{2} + \frac{\sin^{-1} u}{2} + C$$

Substituting in $u = x - 1$, we have

$$\int \sqrt{1-(x-1)^2}dx = \frac{(x-1)\sqrt{1-(x-1)^2}}{2} + \frac{\sin^{-1}(x-1)}{2} + C$$

(c) Calculate the left Riemann sum for a regular partition and $n = 4$. Does the result underestimate or overestimate $\int_0^2 f(x)dx$ (2+1 = 3 pts)?

Solution: The first thing to do is write down the regular partition for the interval $[0, 1]$ and $n = 4$.
The points creating the partition are $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2$ and the intervals of the partition are $[0, \dfrac{1}{2}), [\dfrac{1}{2}, 1), [1, \dfrac{3}{2}), [\dfrac{3}{2}, 2]$.

A Riemann sum for the partition is $\sum_{i=1}^{4} f(y_i)(x_i - x_{i-1})$, where y_i is a point in the closed interval $[x_{i-1}, x_i]$. The left Riemann sum is the Riemann sum where the function is evaluated at the left endpoints, so $y_i = x_{i-1}$ for each i.

Evaluating at these left endpoints, we have $f(0) = \sqrt{1 - (0-1)^2} = 0, f(\frac{1}{2}) = \sqrt{1 - (\frac{1}{2} - 1)^2} = \frac{\sqrt{3}}{2}, f(1) = \sqrt{1 - (1-1)^2} = 1, f(\frac{3}{2}) = \frac{3}{2}$.

Our left Riemann sum is now

$$\sum_{i=1}^{4} f(x_{i-1})(x_i - x_{i-1}) = 0 \cdot (\frac{1}{2} - 0) + \frac{\sqrt{3}}{2} \cdot (1 - \frac{1}{2}) + 1 \cdot (\frac{3}{2} - 1) + \frac{3}{2} \cdot (2 - \frac{3}{2}) = \frac{\sqrt{3} + 5}{4}$$

Looking at the graph we sketched, the function is strictly increasing, so we know that the left Riemann sum will underestimate the integral $\int_0^2 f(x)dx$.

(d) Calculate the right Riemann sum for a regular partition and $n = 4$. Does the result underestimate or overestimate $\int_0^2 f(x)dx$ (2+1 = 3 pts)?

Solution: The partition in this question is the same as in part (c). The right Riemann sum is the Riemann sum where the function is evaluated at the right endpoints, so $y_i = x_i$ for all i.

Evaluating at the right endpoints, we have $f(\frac{1}{2}) = \sqrt{1-(\frac{1}{2}-1)^2} = \frac{\sqrt{3}}{2}$, $f(1) = \sqrt{1-(1-1)^2} = 1$, $f(\frac{3}{2}) = \frac{3}{2}$, $f(2) = 2$.

Our right Riemann sum is now

$$\sum_{i=1}^{4} f(x_i)(x_i - x_{i-1}) = \frac{\sqrt{3}}{2} \cdot (\frac{1}{2} - 0) + 1 \cdot (1 - \frac{1}{2}) + \frac{3}{2} \cdot (\frac{3}{2} - 1) + 2 \cdot (2 - \frac{3}{2}) = \frac{\sqrt{3}+9}{4}$$

The function is strictly increasing, so we know that the right Riemann sum will overestimate the integral $\int_0^2 f(x)dx$.

(e) Evaluate $\int_0^2 f(x)dx$ using a geometric argument. **Hint**: the area of a circle with radius r is πr^2 (2 pts).

Solution: From 0 to 1, the function traces a quarter circle of radius 1, so the area under this portion of the function is a quarter of the area of a circle, $\frac{\pi \cdot 1^2}{4} = \frac{\pi}{4}$.

From 1 to 2, the function traces the line $y = x$. We can split up the area underneath into two parts to make it easy to find the area, namely square with corners $(1, 1), (1, 0), (2, 0), (2, 1)$, which has area 1, and the triangle on top of it with corners $(1, 1), (2, 2), (2, 1)$, which is half of a 1×1 square, so it has area $\frac{1}{2}$. The area under this section is then $\frac{3}{2}$.

(Alternatively, the area under the function from 1 to 2 forms a trapezoid with parallel sides of lengths $a = 1$ and $b = 2$, and height $h = 1$, so using the area formula $A = \frac{a+b}{2} \cdot h$ gives the same result.)

Our total area is then $\frac{\pi + 6}{4}$.

(f) Evaluate $\int_0^2 f(x)dx$ using the fundamental theorem of calculus. **Hint**: split the integral into two integrals (3 pts).

Solution: The idea here is to split up the integral into intervals where we know an antiderivative. From part (b), we know an antiderivative on the interval $[0, 1]$, and on the interval $[1, 2]$, the function is just the line $f(x) = x$, so we know the antiderivative there too.

Working out the first piece,

$$\int_0^1 f(x)dx = \int_0^1 \sqrt{1-(x-1)^2}dx = \left[\frac{(x-1)\sqrt{1-(x-1)^2}}{2} + \frac{\sin^{-1}(x-1)}{2}\right]_0^1$$

$$= \left[\frac{(1-1)\sqrt{1-(1-1)^2}}{2} + \frac{\sin^{-1}(1-1)}{2}\right] - \left[\frac{(0-1)\sqrt{1-(0-1)^2}}{2} + \frac{\sin^{-1}(0-1)}{2}\right] = \frac{\pi}{4}$$

And for the other part, we have

$$\int_1^2 f(x)dx = \int_1^2 xdx = \left[\frac{x^2}{2}\right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

Adding these together, we get $\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx = \frac{\pi+6}{4}$.