Math 105:103 Assignment 1 Solutions

1. (i) Find a function whose derivative is $\frac{1}{x^2} - \frac{2}{x^{5/2}}$.

   Solution: Using the power rule for integrals, a function with derivative $\frac{1}{x^2}$ is $\frac{-1}{x}$.

   Using the power rule for integrals, a function with derivative $\frac{2}{x^{5/2}}$ is $\frac{-4}{3x^{3/2}}$.

   Remembering that the derivative of a sum is the sum of derivatives, we can put this together to get that $\frac{-1}{x} + \frac{4}{3x^{3/2}}$ is a function whose derivative is $\frac{1}{x^2} - \frac{2}{x^{5/2}}$.

   (ii) Find a function whose derivative is $2e^{2t} + 5\sec(3t)\tan(3t)$.

   Solution: Applying the chain rule, the derivative of e^{2t} is $2e^{2t}$, so this will take care of the first summand.

   Recall that the derivative of $\sec(t)$ is $\sec(t)\tan(t)$. Applying the chain rule, the derivative of $\sec(3t)$ is $3\sec(3t)\tan(3t)$, so a function with derivative $5\sec(3t)\tan(t)$ is $\frac{5}{3}\sec(3t)$.

   Now putting this together, we get that the function $e^{2t} + \frac{5}{3}sec(3t)$ is a function with derivative $2e^{2t} + 5\sec(3t)\tan(3t)$.

   (iii) Use parts (a) and (b) to evaluate the indefinite integral

$$\int \left[\frac{1}{s^2} - \frac{2}{s^{5/2}} - 2e^{2s} - 5\sec(3s)\tan(3s)\right] \mathrm{d}s$$

   Solution: Notice that the integrand is just the function from the first problem statement minus the function from the second problem statement. From this (using the property of a derivative of sums), we know a function with derivative $\frac{1}{s^2} - \frac{2}{s^{5/2}} - 2e^{2s} - 5\sec(3s)\tan(3s)$, namely $\frac{-1}{s} + \frac{4}{3s^{3/2}} - e^{2s} - \frac{5}{3}\sec(3s)$.

   This means that the indefinite integral is $\frac{-1}{s} + \frac{4}{3s^{3/2}} - e^{2s} - \frac{5}{3}\sec(3s) + C$.

2. Explain in a few words which change of variable would be appropriate for the following integral, and then use it to evaluate the integral:

$$\int (x^2 - x)(2x^3 - 3x^2 + 14)^{11} \mathrm{d}x$$

Solution: The tricky part of the integrand is $(2x^3 - 3x^2 + 14)^{11}$, so we want a substitution that will make this easier. The derivative of $(2x^3 - 3x^2 + 14)$ is $6x^2 - 6x$, which luckily enough is 6 times the other factor in the integrand.

Now make the substitution $u = 2x^3 - 3x^2 + 14$, which makes $\mathrm{d}u = (6x^2 - 6x)\mathrm{d}x = 6(x^2 - x)\mathrm{d}x$. In order to substitute, need to rewrite this as $\frac{\mathrm{d}u}{6} = (x^2 - x)\mathrm{d}x$, so now we get:

$$\int (x^2 - x)(2x^3 - 3x^2 + 14)^{11} \mathrm{d}x = \int \underbrace{(2x^3 - 3x^2 + 14)}_{u}{}^{11} \underbrace{(x^2 - x)\mathrm{d}x}_{\frac{1}{6}\cdot \mathrm{d}u} = \frac{1}{6} \int u^{11} \mathrm{du}$$

This integral is now easy to evaluate as

$$\frac{1}{6} \int u^{11} \mathrm{du} = \frac{1}{6 \cdot 12} u^{12} + C = \frac{1}{72} u^{12} + C$$

Substituting $u = 2x^3 - 3x^2 + 14$ back into this, we have

$$\int (x^2 - x)(2x^3 - 3x^2 + 14)^{11} \mathrm{d}x = \frac{1}{72}(2x^3 - 3x^2 + 14)^{12} + C$$

3. Find a function g such that

$$g'(x) = \frac{\sin(\ln(x^3))}{4x}$$

How many such functions are there?

Solution: If we can find the antiderivative $\int \frac{\sin(\ln(x^3))}{4x}\mathrm{d}x$, then we'll have such a function.

We want to do a substitution to simplify this integral. We know that the derivative of $\ln(x)$ is $1/x$, so applying the chain rule, the derivative of $\ln(x^3)$ is $\frac{1}{x^3} \cdot 3x^2 = \frac{3}{x}$ (we also could have done this by using the fact that $\ln(x^3) = 3\ln(x)$). This will give us a substitution that clears out the denominator:

Let $u = \ln(x^3)$. Then $\mathrm{d}u = \frac{3}{x}\mathrm{d}x$. To do the substitution, we need to rewrite this as $\frac{1}{3}\mathrm{d}u = \frac{1}{x}\mathrm{d}x$.

Now we have

$$\int \frac{\sin(\ln(x^3))}{4x}\mathrm{d}x = \frac{1}{4} \cdot \frac{1}{3} \int \sin(u)\mathrm{d}u$$

Evaluating gives $\frac{-\cos(u)}{12} + C$, and substituting back in $u = \ln(x^3)$ gives

$$\int \frac{\sin(\ln(x^3))}{4x} = \frac{-\cos(\ln(x^3))}{12} + C$$

Now if we take $g(x) = \frac{-\cos(\ln(x^3))}{12}$, we have $g'(x) = \frac{\sin(\ln(x^3))}{4x}$.
There are infinitely many such functions, since if C is any real number, then $g(x) + C$ also has this property.

Remark: We could have done this problem by doing two changes of variables:

First make the substitution $u = x^3$ to get $\mathrm{d}u = 3x^2\mathrm{d}x$, so $\frac{1}{12} \cdot \frac{1}{u}\mathrm{d}u = \frac{1}{12} \cdot \frac{1}{x^3}\mathrm{d}u = \frac{1}{4x}\mathrm{d}x$.

We now get

$$\int \frac{\sin(\ln(x^3))}{4x}\mathrm{d}x = \frac{1}{12} \int \frac{\sin(\ln(u))}{u}\mathrm{d}u$$

To evaluate the integral on the right, we do another substitution. Let $v = \ln(u)$, so $\mathrm{d}v = \frac{1}{u}\mathrm{d}u$, and we get

$$\int \frac{\sin(\ln(x^3))}{4x}\mathrm{d}x = \frac{1}{12} \int \frac{\sin(\ln(u))}{u}\mathrm{d}u = \frac{1}{12} \int \sin(v)\mathrm{d}v$$

which evaluates to $-\cos(u) + C$, and substituting $v = \ln(u) = \ln(x^3)$ gives the final result

$$\int \frac{\sin(\ln(x^3))}{4x} = \frac{-\cos(\ln(x^3))}{12} + C$$

Remembering the chain rule, this way is completely equivalent to the first method. As we can see from the first solution, making the substitution $u = \ln(x^3)$ right away is more direct and faster.