Math 105 - Practice Midterm 1 for Midterm 2 Solutions

This practice midterm may be harder and/or longer than the real midterm.
Not all question will be worth the same number of points.

1. *Evaluate* $\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$*, or show that it doesn't exist.*
   Let's first evaluate the indefinite integral with the substitution $u = \sqrt{x}$, $2du = \frac{1}{\sqrt{x}}dx$:
   $$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int 2e^{-u}du = -2e^{-u} = -2e^{-\sqrt{x}}.$$
   Then the improper integral is
   $$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{c\to\infty}\left(-2e^{-\sqrt{x}}\right)\Big|_0^c = -2\lim_{c\to\infty}(e^{-\sqrt{c}} - 1) = -2\cdot(0-1) = \boxed{2.}$$

2. *Solve the initial value problem* $y' = \frac{1}{\sqrt{xy}}$*,* $y(1) = 4$*.*
   $$\frac{dy}{dx} = \frac{1}{\sqrt{x}\sqrt{y}} \quad\Rightarrow\quad \int \sqrt{y}dy = \int \frac{1}{\sqrt{x}}dx$$
   $$\Rightarrow\quad \frac{2}{3}y^{3/2} = 2\sqrt{x} + C_1 \quad\Rightarrow\quad y^{3/2} = 3\sqrt{x} + C_2$$
   $$\Rightarrow\quad y = (3\sqrt{x} + C_2)^{2/3}.$$
   $$y(1) = 4 \quad\Rightarrow\quad 4 = (3 + C_2)^{2/3} \quad\Rightarrow\quad 3 + C_2 = 4^{3/2} = 8 \quad\Rightarrow\quad C_2 = 5$$
   $$\Rightarrow\quad \boxed{y = (3\sqrt{x} + 5)^{2/3}.}$$

3. *Find an equation for the plane that is parallel to* $x - 2y + 6z = 1$ *and contains the point* $(4, 0, 2)$*.*
   A plane parallel to that one will have the same normal direction, so we can find an equation for it with the same normal vector $\langle 1, -2, 6\rangle$, of the form
   $$x - 2y + 6z = d.$$
   Then the point $(4, 0, 2)$ determines the d:
   $$4 - 2\cdot 0 + 6\cdot 2 = 16 = d$$
   $$\Rightarrow\quad \boxed{x - 2y + 6z = 16.}$$

4. *Sketch the level curves of* $z = y^2 - \frac{1}{4}x^2$ *at the heights* $z = -1, 0, 1$*.*
   See the last page of these solutions.

5. *Evaluate the limit* $\lim\limits_{(x,y)\to(0,0)} \dfrac{5x - 2y^2}{x + 2y^2}$, *or show that it doesn't exist.*

Plugging in $(0,0)$ would give $\frac{\text{zero}}{\text{zero}}$, but none of the usual tricks seems to simplify this limit. So we should try to use the 2-path test. We can try to approach over a general line $y = mx$:

$$\lim_{(x,y)\to(0,0)} \frac{5x - 2y^2}{x + 2y^2} \xrightarrow{y=mx} \lim_{x\to 0} \frac{5x - 2m^2x^2}{x + 2m^2x^2} = \lim_{x\to 0} \frac{5 - 2m^2x}{1 + 2m^2x} = 5,$$

so over all these lines we get the same value, and we can't use the 2-path test with two of these lines.

However, there is one line that's not included in $y = mx$, namely $x = 0$ (the y-axis; it corresponds to '$m = \infty$'). If we approach over that line, we get

$$\lim_{(x,y)\to(0,0)} \frac{5x - 2y^2}{x + 2y^2} \xrightarrow{x=0} \lim_{y\to 0} \frac{-2y^2}{2y^2} = \lim_{y\to 0} -1 = -1.$$

Hence we can apply the 2-path test with $x = 0$ and for instance $y = 0$: over these two paths we get different values, so the limit $\boxed{\text{does not exist}}$.

Suppose that we didn't think of the possibility $x = 0$. Then we could proceed to try paths of the form $y = mx^a$, and looking at the exponents of x and y in the limit, we should try $y = m\sqrt{x}$, because then

$$\lim_{(x,y)\to(0,0)} \frac{5x - 2y^2}{x + 2y^2} \xrightarrow{y=m\sqrt{x}} \lim_{x\to 0} \frac{5x - 2m^2x}{x + 2m^2x} = \lim_{x\to 0} \frac{5 - 2m^2}{1 + 2m^2} = \frac{5 - 2m^2}{1 + 2m^2}.$$

Hence we get different values for, say, $m = 0$ and $m = 1$, so along the two paths $y = 0$ and $y = \sqrt{x}$, we get different values, and by the 2-path test the limit $\boxed{\text{does not exist}}$.

Note: the above is a detailed explanation; in your solution you do not have to show all of this, you only have to give two paths and show that you get different values for the limit.

6. *Consider the hill given by the function* $z = f(x,y) = \sqrt{1 - x^2 - 4y^2}$.

   (a) *Compute* f_x *and* f_y.

   $$\Rightarrow \quad f_x(x,y) = \frac{-x}{\sqrt{1 - x^2 - 4y^2}}, \qquad f_y(x,y) = \frac{-4y}{\sqrt{1 - x^2 - 4y^2}}$$

   (b) *Find the unit vector that gives the direction of steepest ascent at the point* $\left(\frac{1}{2}, \frac{1}{4}, f\left(\frac{1}{2}, \frac{1}{4}\right)\right)$ *on the hill. Also find a unit vector that gives the direction of no change at that point.*

   The direction of steepest ascent is the direction of the gradient vector evaluated at the point $(x,y) = \left(\frac{1}{2}, \frac{1}{4}\right)$:

$$\nabla f\left(\frac{1}{2}, \frac{1}{4}\right) = \left\langle \frac{-1/2}{\sqrt{1 - \frac{1}{4} - 4 \cdot \frac{1}{16}}}, \frac{-4 \cdot 1/4}{\sqrt{1 - \frac{1}{4} - 4 \cdot \frac{1}{16}}} \right\rangle = \left\langle -\frac{1}{2} \cdot \frac{1}{\sqrt{\frac{1}{2}}}, \frac{-1}{\sqrt{\frac{1}{2}}} \right\rangle = \left\langle -\frac{1}{\sqrt{2}}, -\sqrt{2} \right\rangle.$$

But this is not a unit vector, since its length is

$$\sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{\frac{1}{2}+2} = \sqrt{\frac{5}{2}}.$$

Hence the unit vector that gives the direction of steepest ascent is what we get if we divide the gradient vector by its length:

$$\frac{1}{\sqrt{5/2}}\left\langle -\frac{1}{\sqrt{2}}, -\sqrt{2}\right\rangle = \left\langle -\frac{\sqrt{2}}{\sqrt{5}}\cdot\frac{1}{\sqrt{2}}, -\frac{\sqrt{2}}{\sqrt{5}}\cdot\sqrt{2}\right\rangle = \boxed{\left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\rangle}.$$

The direction of no change is given by either of the two unit vectors orthogonal to the one above:

$$\boxed{\left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle \quad \text{or} \quad \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\rangle.}$$

Here we use the fact that the two unit vectors orthogonal to a unit vector $\langle a, b\rangle$ are $\langle b, -a\rangle$ and $\langle -b, a\rangle$.

(c) *Suppose you're walking over the hill along the path that is right above the path $(x(t), y(t)) = (t, t^2)$ in the xy-plane. As you pass the point $\left(\frac{1}{2}, \frac{1}{4}, f\left(\frac{1}{2}, \frac{1}{4}\right)\right)$, at what rate is your height changing?*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(1/2, 1/4)\cdot\left(\frac{d}{dt}t\right)\Big|_{t=1/2} + f_y(1/2,1/4)\cdot\left(\frac{d}{dt}t^2\right)\Big|_{t=1/2}$$

$$= -\frac{1}{\sqrt{2}}\cdot 1 + (-\sqrt{2})\cdot\left((2\cdot\frac{1}{2}\right) = \boxed{-\frac{1}{\sqrt{2}} - \sqrt{2}}\left(= -\frac{3}{\sqrt{2}}\right).$$

7. *Find the critical points of $f(x,y) = \frac{1}{2}x^2 + 4xy + y^3 + 8y^2 + 3x + 2$, and classify each one as a maximum, minimum or saddle point.*

$$f_x = x + 4y + 3 = 0, \qquad f_y = 4x + 3y^2 + 16y = 0$$

To solve these equations, we can for instance isolate x from the first equation, and plug that into the second:

$$x = -4y - 3 \;\Rightarrow\; 4(-4y-3) + 3y^2 + 16y = 0$$
$$\Rightarrow\; 3y^2 - 12 = 0 \;\Rightarrow\; y^2 = 4 \;\Rightarrow\; y = \pm 2.$$

Plugging $y = 2$ into $x = -4y - 3$ gives $x = -11$, and for $y = -2$ we get $x = 5$. So the critical points are $\boxed{(-11, 2)}$ and $\boxed{(5, -2)}$.

To classify them using the Second Derivative Test, we need the second partial derivatives and the discriminant:

$$f_{xx} = 1, \quad f_{yy} = 6y + 16, \quad f_{xy} = 4$$
$$\Rightarrow D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 1\cdot(6y+16) - 4^2 = 6y.$$

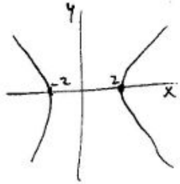
Then for the critical point $(-11, 2)$ we have $D(-11, 2) = 12 > 0$, so we need to look at $f_{xx}(-11, 2) = 1 > 0$, which tells us that $\boxed{(-11,2) \text{ is a local minimum}}$.

For $(5, -2)$ we have $D(5, -2) = -12 < 0$, so $\boxed{(5,-2) \text{ is a saddle point}}$.

4| Level curves of $y^2 - \frac{1}{4}x^2 = z$ for $z = -1, 0, 1$

• $z = -1 \Rightarrow y^2 - \frac{1}{4}x^2 = -1 \Rightarrow$ hyperbola:

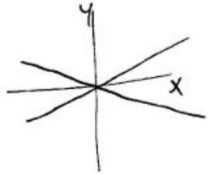
$\left(y = 0 \Rightarrow -\frac{1}{4}x^2 = -1 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \right)$



• $z = 0 \Rightarrow y^2 - \frac{1}{4}x^2 = 0$

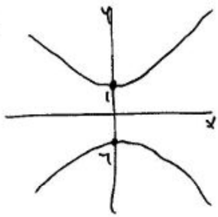
$(y - \frac{1}{2}x)(y + \frac{1}{2}x) \Rightarrow y = \frac{1}{2}x$ or $y = -\frac{1}{2}x$

$\Rightarrow$ lines:



• $z = 1 \Rightarrow y^2 - \frac{1}{4}x^2 = 1 \Rightarrow$ hyperbola:

$\left(x = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1 \right)$



The level curves together: