Math 421/510, Spring 2009
Homework Set 4 and Take-home Final
due on April 20

Instructions

- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, state the result clearly with either a reference or a self-contained proof.

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1. Here is an application of the notion of weak/weak* convergence in probability theory.
   (a) Prove Helly's selection principle. Namely, let $\{\mu_n\}$ be a sequence of probability measures on $[0, 1]$. Then there exists a probability measure μ and a subsequence $\{\mu_{n_k} : k \geq 1\}$ such that for all $f \in C[0, 1]$

$$\int f d\mu_{n_k} \to \int f \, d\mu \quad \text{as } k \to \infty.$$

   (b) Given any sequence of numbers $\{a_n : n \in \mathbb{Z}\}$, how can we determine whether these numbers occur as the Fourier coefficients of some probability measure on $[-\pi, \pi]$? The key idea here is positive definiteness.
   A doubly infinite sequence $\{a_m : m \in \mathbb{Z}\}$ of complex numbers is said to be a positive definite sequence if for each $n = 1, 2, \cdots$, the $n \times n$ matrix $A_n = ((a_{i-j}))$, $0 \leq i, j \leq n-1$ constructed from this sequence is positive semidefinite, i.e., for all $N \geq 1$ and all $z \in \mathbb{C}^N$,

$$\sum_{n,m=1}^{N} a_{n-m} z_n \bar{z}_m \geq 0.$$

   Show that if μ is a probability measure, then the sequence

$$a_n = \int e^{-inx} d\mu(x)$$

   is positive definite, in the sense described above.
   (c) Prove the converse of the statement in part (b), originally due to Herglotz. More precisely, let $\{a_n : n \in \mathbb{Z}\}$ be a positive definite sequence and suppose $a_0 = 1$. Then show that there exists a

probability measure μ on $[-\pi, \pi]$ such that

$$a_n = \int_{-\pi}^{\pi} e^{-inx} d\mu(x).$$

2. We have seen that for a convex set K in a Banach space X, the norm closure of K equals the weak closure of K. Is this statement always true if K is a convex subset of $U = X^*$ (for some Banach space X) and "weak" is replaced by "weak*"?
3. Is $U = X^*$ equipped with the weak* topology metrizable?
4. Show that every normed linear space X is isometric to a subspace of $C(K)$ for some compact Hausdorff space K. If X is separable, show that K can be chosen to be a compact metric space.
5. The previous exercise suggests that the spaces $C(K)$ (of continuous functions on a given compact Hausdorff space K) deserve special attention, containing as they do isometric copies of every normed linear space. Let us catalog some important properties of the spaces $C(K)$ for different choices of K.

   Remember the Cantor middle-third set (we will denote this by Δ)? The goal of this problem is to uncover the "universal" nature of $C(\Delta)$. More precisely, we will prove that $C(\Delta)$ is the "biggest" among the spaces $C(K)$, where K is a compact metric space. We will do this by showing that every compact metric space K is the continuous image of Δ.
   (a) Convince yourself that the statement above is right, i.e., if $\varphi : \Delta \to K$ is a continuous surjection, then there exists a linear isometry from $C(K)$ to $C(\Delta)$.
   (b) Show that $[0, 1]$ is a continuous image of Δ, as is the cube $[0, 1]^{\mathbb{N}}$.
   (c) Recalling that the elements of Δ are sequences of 0-s and 2-s, i.e., $\Delta = \{0, 2\}^{\mathbb{N}}$, let us endow Δ with the following natural metric:

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{3^n},$$

   where $\{a_n\}$ and $\{b_n\}$ are the sequences of digits (0-s and 2-s) occurring in the ternary expansion of x and y respectively. Convince yourself (but you need not submit a solution) that d is equivalent to the usual metric on Δ. Moreover, d has the additional property that $d(x, y) = d(x, z)$ implies that $y = z$. In subsequent discussions, take the metric on Δ to be the one described above.
   Show that every compact metric space is homeomorphic to a closed subspace of $[0, 1]^{\mathbb{N}}$.

(d) Deduce that every compact metric space K is the continuous image of Δ. In light of part (a) of this problem, we now know that $C(K)$ is isometric to a closed subspace of $C(\Delta)$.

6. OK, we have just now seen that $C(\Delta)$ is "universal" for the class of spaces $C(K)$, for compact metric spaces K. In particular, $C[0,1]$ sits inside $C(\Delta)$! This line of argument must seem backward, given how much more publicity $C[0,1]$ receives than $C(\Delta)$ (think of your first course in analysis). Let's ask whether $C[0,1]$ can be universal too, i.e., whether $C(\Delta)$ embeds isometrically into $C[0,1]$.
   (a) Given a function $f \in C(\Delta)$, define an extension $\tilde{f}$ of f as a continuous function on $[0,1]$ so that the extension map $E(f) = \tilde{f}$ from $C(\Delta)$ into $C[0,1]$ is a linear isometry.
   (b) Deduce that $C(\Delta)$ is isometric to a complemented subspace of $C[0,1]$. (A subspace M is complemented in a Banach space X if M is the range of a continuous linear projection P on X).
   (c) Pull together the facts compiled in problems 4-6 to prove the Banach-Mazur theorem: every separable normed linear space is isometric to a subspace of $C[0,1]$.
7. Prove the spectral radius formula stated in class.
8. Find the spectra of the left shift operator $L : \ell^2 \to \ell^2$ defined by

$$L(a_0, a_1, a_2, \cdots) = (a_1, a_2, \cdots),$$

   and the Volterra operator $V : C[0,1] \to C[0,1]$ defined by

$$Vx(s) = \int_0^s x(r)\, dr.$$