

Math 421/510, Spring 2009, Homework Set 1

(due on Friday January 30)

Instructions

- Homework will be collected at the end of lecture on Friday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, state the result clearly with either a reference or a self-contained proof.

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1. Follow the sketch of proof given in class to prove that any norm in a normed linear space obeying the parallelogram law arises from a unique inner product.
 2. What is the orthogonal complement of $\ell^1(\mathbb{N})$ considered as a subspace of $\ell^2(\mathbb{N})$?
 3. Show that the closed unit ball (hence any closed ball) in a normed linear space X is compact if and only if X is finite-dimensional.
 4. In our discussion of Hilbert spaces, we saw that given any closed convex nonempty subset K of a Hilbert space \mathbb{H} , and $h \in \mathbb{H}$, there exists a vector $k_0 \in K$ such that $\|h - k_0\| = \text{dist}(h, K)$. The point of this problem is to investigate whether this result extends to more general Banach space settings. The key concept turns out to be something called *uniform convexity*, due to Clarkson.

Definition. A normed linear space X is said to be *uniformly convex* if there exists an increasing function $\epsilon : (0, \infty) \rightarrow (0, \infty)$, $\epsilon(r) \rightarrow 0$ as $r \rightarrow 0+$, such that for all $x, y \in X$ with $\|x\|_X, \|y\|_X \leq 1$, the following inequality holds.

$$\left\| \frac{x + y}{2} \right\|_X \leq 1 - \epsilon(\|x - y\|_X).$$

- (a) Verify that every Hilbert space is uniformly convex.
- (b) But there are plenty of non-Hilbert uniformly convex spaces as well. A famous result of Clarkson says that L^p (of any measure space) is uniformly convex for $1 < p < \infty$. Prove Clarkson's result in the simpler case $2 \leq p < \infty$. (Hint: Given $2 \leq p < \infty$,

and real numbers a and b ,

$$|a + b|^p + |a - b|^p \leq 2^{p-1}(|a|^p + |b|^p).$$

- (c) Prove the analogue of our Hilbert space result for uniformly convex spaces. Namely, let X be a uniformly convex Banach space. Let K be a closed convex subset of X , z any point of X . Then there is a unique point y of K which is closer to z than any other point in K . (Hint: Mimic as much of the Hilbert space proof as possible.)
- (d) Show that $C[-1, 1]$ equipped with the sup norm is not uniformly convex. In fact, the sup norm is not even strictly subadditive on the unit sphere, which is a weaker notion than uniform convexity. (A norm $\|\cdot\|_X$ on a normed linear space X is said to be strictly subadditive on $S \subseteq X$ if $\|x + y\|_X < \|x\|_X + \|y\|_X$ for any $x, y \in S$ that are not non-negative multiples of each other.)
- (e) Show that the conclusion of part (c) fails in $C[-1, 1]$, by working out the following example. Take K to be the subset of $C[-1, 1]$ consisting of functions k that satisfy

$$\int_{-1}^0 k(t) dt = 0, \quad \int_0^1 k(t) dt = 0.$$

Check that K is convex (in fact a linear subspace), and closed. Take for $z(t)$ any function in $C[-1, 1]$ satisfying

$$\int_{-1}^0 z(t) dt = 1, \quad \int_0^1 z(t) dt = -1.$$

Show that there is no point closest to z in K .