

Math 421/510, Spring 2008
Homework Set 5, due Friday April 18

Instructions

- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, state the result clearly with either a reference or a self-contained proof.

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1. Fill in the details of the proof of the Riesz Representation Theorem that were left as exercises in class.
 2. The Hahn-Banach theorem that we proved in class was for normed linear spaces over the real field. Prove the theorem when the underlying scalar field is \mathbb{C} .
 3. An important application of the Hahn-Banach theorem is to prove separation theorems for convex sets. We discuss one such problem in this exercise.

Let X be a linear space over the reals, and $K \subseteq X$ a convex set that contains the origin as an interior point. We denote the *gauge* p_K of K with respect to the origin as follows:

$$p_K(\mathbf{x}) = \inf \left\{ a > 0 : \frac{\mathbf{x}}{a} \in K \right\}.$$

Observe that $p_K(\mathbf{x}) < \infty$ for every \mathbf{x} (why?).

- (a) Show that p_K is positive homogeneous and subadditive.
- (b) Let K be a nonempty convex subset of a linear space X over the reals; suppose that all points of K are interior. Show that any point y not in K can be separated from K by a hyperplane $\ell(\mathbf{x}) = c$; i.e., there is a linear functional depending on y such that

$$\ell(\mathbf{x}) < c \quad \text{for all } x \in K; \quad \ell(\mathbf{y}) = c.$$

This is known as the *hyperplane separation theorem*.

- (c) Prove the following generalization of hyperplane separation: X is a linear space over \mathbb{R} ; H and M are disjoint convex subsets of X , at least one of which has an interior point. Then H and M can be separated by a hyperplane $\ell(\mathbf{x}) = c$; i.e., there is a

nonzero linear functional ℓ and a number c such that

$$\ell(\mathbf{u}) \leq c \leq \ell(\mathbf{v})$$

for all $\mathbf{u} \in H$, $\mathbf{v} \in M$.

4. Given a Schauder basis $\{\mathbf{x}_n\}$ of a Banach space X , recall the projection maps

$$P_n \mathbf{x} = \sum_{i=1}^n a_i \mathbf{x}_i \quad \text{where} \quad \mathbf{x} = \sum_{i=1}^{\infty} a_i \mathbf{x}_i.$$

Show that every P_n is continuous, and in fact $\sup_n \|P_n\| < \infty$.

5. Prove the following “test” for basic sequences, due to Banach. A sequence $\{\mathbf{x}_n\}$ of nonzero vectors is a (Schauder) basis for the Banach space X if and only if
- (i) $\{\mathbf{x}_n\}$ has dense linear span in X , and
 - (ii) there is a constant K such that

$$\left\| \sum_{i=1}^n a_i \mathbf{x}_i \right\| \leq K \left\| \sum_{i=1}^m a_i \mathbf{x}_i \right\|$$

for all scalars $\{a_i\}$ and all $n < m$.

6. Prove that a closed linear subspace Y of a reflexive Banach space is reflexive.
7. (a) We have seen in class that in ℓ^1 , a sequence is weakly convergent if and only if it is norm convergent. Use this to show that the weak topology on the unit ball in ℓ^1 is not metrizable.
- (b) Show that weak convergence does not imply strong convergence for ℓ^p , $1 < p < \infty$.
8. Let $B = \text{ball } M[0, 1]$, and define

$$d(\mu, \nu) = \sum_{n=0}^{\infty} 2^{-n} \left| \int_0^1 x^n d\mu - \int_0^1 x^n d\nu \right|, \quad \text{for } \mu, \nu \in M[0, 1].$$

Show that d is a metric that defines the weak-star topology on B but not on $M[0, 1]$.

9. Assume that X is a Banach space such that X^* is separable. If K is a weakly compact subset of X , show that K with the relative weak topology is metrizable.