Theorem. If X is a metrizable, locally compact space that is σ-compact, show that $C_0(X)$ is separable.

Proof. The theorem will be proved using the following sequence of lemmas.

Lemma 1. *Given any locally compact metric space Y, and $y \in Y$, there exists $\delta_y > 0$ such that $B(y;\delta_y)$ has compact closure.*

Proof. By the definition of local compactness, every $y \in Y$ has a compact neighborhood, which (by definition of a neighborhood) contains an open ball centered at y. □

Lemma 2. *There exists a countable collection $\mathcal{L} = \{L_m : m \in \mathbb{N}\}$ of compact sets in X with the following property. For any compact set $K \subseteq X$, there exists $L_m \in \mathcal{L}$ such that $K \subseteq L_m$.*

Proof. Since X is σ-compact (i.e. a countable union of compact sets), every open cover of X must have a countable subcover. Since the family of balls $\{B(x;\delta_x) : x \in X\}$, where δ_x is as in Lemma 1, is an open cover for X, there exists a countable subcollection $\{x_i : i \geq 1\} \subseteq X$ such that

$$X \subseteq \bigcup_{i=1}^{\infty} B\big(x_i;\delta_{x_i}\big).$$

Let $\mathcal{L}$ denote the collection of sets consisting of all possible finite unions of the closed balls $\overline{B(x_i;\delta_{x_i})}$. By Lemma 1, each element of $\mathcal{L}$ is a compact subset of X (recall that a finite union of compact sets is compact). Moreover, $\mathcal{L}$ is countable, since the collection of finite subsets of a countable set is countable. Finally, every compact $K \subseteq X$ is contained in a finite union of the $\{B(x_i;\delta_{x_i}) : i \geq 1\}$, and hence is contained in some member of $\mathcal{L}$. □

Construction of a countable dense subset of $C_0(X)$

Lemma 3. *For any $n \in \mathbb{N}$, there exists a countable covering $\mathcal{U}^{(n)}$ of X by relatively compact open balls with diameter bounded above by $\frac{1}{n}$.*

Proof. Since $\{B(x;\min\{\delta_x,\frac{1}{n}\}) : x \in X\}$ forms an open cover of X, by σ-compactness we can extract a countable subcover with the desired properties. □

For every $n, m \in \mathbb{N}$, let $\mathcal{U}^{(n,m)} = \{B_1^{(n,m)}, B_2^{(n,m)}, \cdots, B_{J_{n,m}}^{(n,m)}\} \subseteq \mathcal{U}^{(n)}$ be a finite subcover of L_m, where L_m is as in Lemma 2. While there may be many choices for $\mathcal{U}^{(n,m)}$, we fix one for every choice of (n,m), and only work with these in the sequel. Applying the partition of unity lemma, there exists a finite collection of functions $\{\chi_j^{(n,m)} : 1 \leq j \leq J_{n,m}\}$ with the following properties: for every $1 \leq j \leq J_{n,m}$,

(i) the function $\chi_j^{(n,m)} : X \to [0,1]$ is continuous,
(ii) $\operatorname{supp}(\chi_j^{(n,m)}) \subseteq \overline{B}_j^{(n,m)}$, and
(iii) $\sum_j \chi_j^{(n,m)}(x) \equiv 1$ for all $x \in L_m$.

We now define

$$\mathfrak{X}_{n,m} = \left\{ \sum_{j=1}^{J_{n,m}} a_j \chi_j^{(n,m)} : a_j \in \mathbb{Q} \quad \forall j \right\} \quad \text{and} \quad \mathfrak{X} = \bigcup_{n,m} \mathfrak{X}_{n,m}.$$

By construction, $\mathfrak{X}_{n,m}$ is countable for every $n, m \in \mathbb{N}$, therefore so is $\mathfrak{X}$. Further every function in $\mathfrak{X}_{n,m}$ is continuous and has support in the compact set $\bigcup_{j=1}^{J_{n,m}} \overline{B}_j^{(n,m)}$, which implies that $\mathfrak{X} \subseteq C_0(X)$. $\square$

It therefore remains to prove the following.

Lemma 4. *$\mathfrak{X}$ is dense in $C_0(X)$.*

Proof. Fix $f \in C_0(X)$ and $\epsilon > 0$. Since f is uniformly continuous (why?), there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{4} \text{ whenever } d(x,y) < \delta. \tag{1}$$

Let $F_\epsilon = \{x \in X : |f(x)| > \epsilon\}$, so that F_ϵ is open in X with compact closure. Then $\overline{F}_{\frac{\epsilon}{2}} \subseteq F_{\frac{\epsilon}{4}}$, which implies that $\overline{F}_{\frac{\epsilon}{2}}$ and $F^c_{\frac{\epsilon}{4}}$ are disjoint closed sets. Since the map $x \mapsto d(x, F^c_{\frac{\epsilon}{4}})$ is continuous on X and strictly positive on $F_{\frac{\epsilon}{4}}$, and $\overline{F}_{\frac{\epsilon}{2}}$ is compact, therefore

$$d = \operatorname{dist}(\overline{F}_{\frac{\epsilon}{2}}, F^c_{\frac{\epsilon}{4}}) > 0. \quad \text{Similarly } d' = \operatorname{dist}(\overline{F}_{\frac{\epsilon}{4}}, F^c_{\frac{\epsilon}{8}}) > 0 \tag{2}$$

Choose n large enough to that $1/n < \min(\delta, d, d')$, where δ and (d, d') have been defined in (1) and (2) respectively. By Lemma 2, we can pick an integer m such that $\overline{F}_{\frac{\epsilon}{16}} \subseteq L_m$. Set

$$g = \sum_{j=1}^{J_{n,m}} a_j \chi_j^{(n,m)}, \text{ where } \begin{cases} |a_j - f(c_j)| < \frac{\epsilon}{4},\ a_j \in \mathbb{Q} & \text{if } c_j \in F_{\frac{\epsilon}{4}} \\ a_j = 0 & \text{otherwise.} \end{cases} \tag{3}$$

Here c_j denotes the center of the ball $B_j^{(n,m)}$. Then

$$\operatorname{supp}(g) \subseteq \overline{F}_{\frac{\epsilon}{8}}. \tag{4}$$

This is because for $x \in B_j^{(n,m)} \setminus F_{\frac{\epsilon}{8}}$, our choice of n implies $c_j \notin \overline{F}_{\frac{\epsilon}{4}}$, so that $a_j = 0$ by (3). Therefore $g \equiv 0$ on $F^c_{\frac{\epsilon}{8}}$.

We claim that $\sup_{x \in X} |f(x) - g(x)| < \epsilon$. To see this, suppose first that $x \notin F_{\frac{\epsilon}{8}}$. Then by the support property (4) of g, $|f(x) - g(x)| = |f(x)| \leq \frac{\epsilon}{8} < \epsilon$. If $x \in F_{\frac{\epsilon}{8}}$, by (iii) on page 1,

$$\begin{aligned}
|f(x) - g(x)| &= |\sum_j (f(x) - a_j)\chi_j^{(n,m)}(x)| \\
&\leq \sum_j |f(x) - a_j| \chi_j^{(n,m)}(x) \\
&\leq \sum_{j : c_j \in F_{\frac{\epsilon}{4}}} (|f(x) - f(c_j)| + |f(c_j) - a_j|)\chi_j^{(n,m)}(x) \\
&\qquad\qquad + \sum_{j : c_j \notin F_{\frac{\epsilon}{4}}} |f(x)| \chi_j^{(n,m)}(x) \\
&< \left(\frac{\epsilon}{4} + \frac{\epsilon}{4}\right) + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

In the last two steps of the above calculation we have used the description of a_j-s from (3), Lemma 3 and (1). The last step also uses the fact that if $c_j \notin F_{\frac{\epsilon}{4}}$ and $1/n < d$, then (by (2)) for any $x \in B_j^{(n,m)}$, $x \notin \overline{F}_{\frac{\epsilon}{2}}$, so that $|f(x)| < \frac{\epsilon}{2}$. $\square$