Math 421/510, Spring 2008, Final Exam
(Due date: Monday April 21)

Instructions

- The final exam should be submitted to the instructor's mailbox by 5 pm on Monday April 21. **There will be no extensions for the final.**
- Unlike homework assignments, you must work on the final on your own. If you need hints or clarifications, please feel free to talk to the instructor.
- Answers should be clear, legible, and in complete English sentences. Solutions must be self-contained – only results proved in class can be used without proof.

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1. Let $\mathfrak{X}$ be a closed subspace of the Hilbert space $L^2[0,1]$, and assume that every element of $\mathfrak{X}$ is essentially bounded (i.e., belongs to $L^\infty[0,1]$). Prove that $\dim \mathfrak{X} < \infty$.
2. An important class of bounded linear operators on Hilbert spaces that arises in the study of integral equations is the class of compact operators. Given two Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, an operator $K : \mathbb{H}_1 \to \mathbb{H}_2$ is said to be *compact* if the image under K of the unit ball in $\mathbb{H}_1$ has compact closure in $\mathbb{H}_2$. Let $\mathcal{K}(\mathbb{H}_1, \mathbb{H}_2)$ denote the class of compact operators.
   (a) Show that $\mathcal{K}(\mathbb{H}_1, \mathbb{H}_2)$ is a closed subset of $\mathcal{B}(\mathbb{H}_1, \mathbb{H}_2)$.
   (b) Given a complex valued function $a(t)$ which is continuous on $[a,b]$, let $A : L^2[a,b] \to L^2[a,b]$ be the bounded linear operator given by
   $$(Af)(t) = a(t)f(t).$$
   Characterize all functions $a \in C[a,b]$ for which A is compact.
3. Let $\mathfrak{X}$ be a normed space. Recall that the *convex hull* of any subset of $\mathfrak{X}$ is the smallest convex set containing the given set. Show that the norm closure of the convex hull of any subset S of $\mathfrak{X}$ contains the weak sequential closure of S. (An equivalent way of formulating this result is as follows: Let $\{\mathbf{x}_n\}$ be a sequence in a normed vector space $\mathfrak{X}$ that converges weakly to $\mathbf{x}$. Show that for every $\epsilon > 0$ and $m \in \mathbb{N}$, there is a convex combination $\mathbf{y} = \sum_{n \geq m} \lambda_n \mathbf{x}_n$, i.e., a finite sum with non-negative coefficients and $\sum_{n \geq m} \lambda_n = 1$, such that $||\mathbf{x} - \mathbf{y}|| < \epsilon$.)