Math 320 Final Exam Practice Problems

Instructions

(i) Final solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) Self-contained solutions are optimal. If in doubt whether to include the proof of a statement, ask your instructor.

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1. Let $S \subset \mathbb{R}^n$. Define

$$T = \{x \in S\colon \text{ for every } r > 0,\ B(x,r) \cap S \text{ is uncountable}\}.$$

   Prove that $S \backslash T$ is countable.

2. Let X be a set. Let $\mathcal{F}$ be the set of functions $f\colon X \to \{0,1\}$. Prove that $\mathcal{F}$ is either finite or uncountable.

3. Define $\ell^\infty(\mathbb{R})$ to be the set of all infinite sequences $(x_1, x_2, \ldots)$ of real numbers for which $\sup\{x_i\}$ is finite. Define an order $<$ on ℓ^∞ as follows: If $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2)$, we say $x < y$ if $x_1 < y_1$, or if $x_1 = y_1$ and $x_2 < y_2$, or if $x_1 = y_1,\ x_2 = y_2,\ x_3 < y_3$, etc.

   (i) Prove that $(\ell^\infty(\mathbb{R},\ <)$ is an ordered set.

   (ii) Does $(\ell^\infty(\mathbb{R},\ <)$ have the least upper bound property? Prove that your answer is correct.

4. (i) Define L to be the set of functions $f\colon [0,1] \to \mathbb{R}$ satisfying $f(0) = 0$, and $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [0,1]$. Let $\{f_n\}$ be a sequence in L, and define

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x).$$

   Prove that $F \in L$.

   (ii) Define M to be the set of continuous functions $g\colon [0,1] \to \mathbb{R}$ satisfying $f(0) = 0$. Let $\{g_n\}$ be a sequence in M, and define

$$G(x) = \sum_{n=1}^{\infty} 2^{-n} g_n(x).$$

   Must it be true that $G \in M$? If so, prove it. If not, provide a counter-example and prove that your example is correct.

5. Let $(\mathbb{R}, +, \cdot)$ be the field of real numbers. Let $\mathbb{R} \cup \{p\}$ be the set obtained by adding one additional element to the set of real numbers. Prove that the operations $+$ and $\cdot$ cannot be extended to $\mathbb{R} \cup \{p\}$ to make $\mathbb{R} \cup \{p\}$ a field.

6. Prove that every open set $U \subset \mathbb{R}^n$ can be written as a *countable* union of open neighborhoods $N_r(x)$.

7. A set $E \subset \mathbb{R}^n$ is called a F_σ set if it can be written as a countable union of closed sets. Give an example of a F_σ set that is neither open nor closed. Prove that your example is correct.

8. Let $f : [0,1] \to [0,1] \times [0,1]$ be continuous and one-to-one.

   (a) Show that f cannot be onto.

   (b) Moreover, show that the range of f is nowhere dense in $[0,1] \times [0,1]$.

9. Determine whether the following statement is true or false, with adequate justification: *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and open, it is strictly monotone.* Recall that a map is open if it maps open sets to open sets.

10. Let M be a compact metric space, and let $f : M \to M$ satisfy

$$d(f(x), f(y)) \geq d(x,y) \quad \text{for all } x, y \in M.$$

   Prove that

   (a) f is an isometry, i.e., $d(f(x), f(y)) = d(x,y)$ for all $x, y \in M$.

   (b) f is onto.

11. Let $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ be normed vector spaces over $\mathbb{R}$, and let $T : V \to W$ a linear map, i.e.,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \text{ for } x, y \in V, \quad \alpha, \beta \in \mathbb{R}.$$

   Show that the following are equivalent:

   (i) T is Lipschitz.

   (ii) T is uniformly continuous.

   (iii) T is continuous everywhere.

   (iv) T is continuous at $0 \in V$.

   (v) There is a constant $C < \infty$ such that

$$||T(v)||_W \leq C||v||_V \text{ for all } v \in V.$$

   We define the *norm of T, denotes* $||T||$ to be the smallest constant C that works in part (v).

12. Fix $y \in \mathbb{R}^n$ and define a linear map $L : \mathbb{R}^n \to \mathbb{R}$ by $L(x) = x \cdot y$. Show that L is continuous and compute the norm of L.

13. Show that the definite integral

$$I(f) = \int_a^b f(t)\, dt$$

   is continuous from $C[a,b]$ into $\mathbb{R}$. What is $||I||$?

14. Determine whether the following statement is true or false: any two norms $||\cdot||_a$ and $||\cdot||_b$ on a finite-dimensional vector space V are equivalent, i.e., there exist constants C_1 and C_2 such that

$$C_1||x||_a \leq ||x||_b \leq C_2||x||_a \quad \text{for all } x \in V.$$

15. Give an example of a vector space with two non-equivalent norms. Explain your answer.

Solutions

1. For each $x \in S\backslash T$, there exists $r > 0$ so that $B(x,r) \cap S$ is countable. Select a rational number $r_x \in (r/2, r) \cap \mathbb{Q}$, and a rational point $y_x \in B(x, r/2) \cap \mathbb{Q}^n$.

   Note that $x \in B(y_x, r_x)$, and $B(y_x, r_x) \subset B(x,r)$, so $B(y_x, r_x) \cap S$ is countable. Define $A = \{(r_x, y_x) \colon x \in S\}$ (note that there could be several distinct $x \in S$ which get mapped to the same pair (r_x, y_x), but A is a set, so it contains each element at most once). Since $A \subset \mathbb{Q}^{n+1}$, A is countable.

   Now, since $x \in B(y_x, r_x)$, $x \in \bigcup_{(y_x,r_x)\in A} B(y_x, r_x)$, so $S\backslash T \subset \bigcup_{(y_x,r_x)\in A} B(y_x, r_x) \cap S$. The latter is a countable union of countable sets, so it is countable. We conclude that $S\backslash T$ is countable.

2. If X is finite, then $|\mathcal{F}| = 2^{|X|}$, which is finite. If X is infinite, then let $\phi \colon \mathbb{N} \to X$ be an injection. Let $\mathcal{G}$ be the set of functions $g \colon \mathbb{N} \to \{0,1\}$; clearly $\mathcal{G}$ is uncountable, since it is in one-to-one correspondence with the set of infinite binary strings (indeed, each function $g \colon \mathbb{N} \to \{0,1\}$ corresponds to the binary string $(g(1), g(2), g(3), \ldots)$.). Then the map from $\mathcal{G}$ to $\mathcal{F}$ which sends the function $g \colon \mathbb{N} \to \{0,1\}$ to the function

$$f(x) = \begin{cases} g(\phi^{-1}(x)),\ x \in \phi(\mathbb{N}), \\ 0,\ x \notin \phi(\mathbb{N}) \end{cases}$$

   is an injection. Since $\mathcal{G}$ is uncountable, we conclude that $\mathcal{F}$ is uncountable as well.

3. (i) First, we will show that for all $x, y \in \ell^\infty(\mathbb{R}$, precisely one of $x < y$, $x > 0$, or $x = y$ holds. Suppose $x \neq y$. Let k be the smallest index for which $x_k \neq y_k$. Since $\mathbb{R}$ is ordered, we must have either $x_k < y_k$ or $y_k < x_k$. If the former holds then $x < y$, while if the latter holds then $y < x$.

   Next, suppose $x < y$ and $y < z$. Let k be the smallest index for which $x_k \neq y_k$, and let ℓ be the smallest index for which $y_\ell \neq z_\ell$; we have $x_k < y_k$ and $y_\ell < z_\ell$. If $k \leq \ell$, then $x_i = y_i = z_i$ for all $i < k$, and $x_k < y_k \leq z_k$, so $x < z$. If instead $k > \ell$, then then $x_i = y_i = z_i$ for all $i < \ell$, and $x_\ell = y_\ell < z_\ell$, so $x < z$. We conclude that $x < z$.

   (ii) For each $k \in \mathbb{N}$, define $p_n = (x_1, x_2, \ldots)$, where $x_i = i$ for $1 \leq i \leq n$ and $x_i = 0$ for all $i > n$. We have that $p_n \in \ell^\infty(\mathbb{R})$ for each n. Define $E = \{p_n \colon n \in \mathbb{N}\}$. Then E is bounded above (indeed, the element $(2, 0, 0, \ldots,)$ is an upper bound for E). However, E does not have a least upper bound. Indeed, suppose that $y = (y_1, y_2, \ldots)$ is the least upper bound for E. We must have $y_1 \geq 1$. Let k be the smallest index so that $y_k \neq k$. Since $y > p_{k+1})$, we must have $y_k > k$. But it's easy to verify that the element $(1, 2, 3, \ldots, k-1, \frac{y_k+k}{2})$ is also an upper bound for E, and this element is smaller than y; thus y is not a least upper bound for E.

   We conclude that if $y = (y_1, y_2, \ldots)$ is an upper bound for E, then $y_k = k$ for each $k \in \mathbb{N}$. But this sequence is not an element of $\ell^\infty(\mathbb{R})$, since $\sup\{y_k\}$ is not finite. We conclude that this set E does not have a least upper bound, and thus $(\ell^\infty(\mathbb{R}), <)$ does not have the least upper bound property.

4. (i) $F(0) = \sum_{n=1}^\infty 2^{-n} f(0) = \sum_{n=1}^\infty 2^{-n} 0 = 0$. Next, observe that for each index n, we have $|f(x)| \leq |x| \leq 1$ for all $x \in [0,1]$. Thus $\sum_{n=1}^\infty |2^{-n} f(x)| \leq \sum_{n=1}^\infty |2^{-n}| \leq 1$, so $\sum_{n=1}^\infty f_n(x)$ is absolutely convergent, and similarly $\sum_{n=1}^\infty f_n(y)$ is absolutely

convergent. Thus

$$\begin{aligned}
F(x) - F(y) &= \Big|\sum_{n=1}^{\infty} 2^{-n} f_n(x) - \sum_{n=1}^{\infty} 2^{-n} f_n(y)\Big| \\
&= \Big|\sum_{n=1}^{\infty} 2^{-n}(f_n(x) - f_n(y))\Big| \\
&\le \sum_{n=1}^{\infty} 2^{-n}|f_n(x) - f_n(y)| \\
&\le \sum_{n=1}^{\infty} 2^{-n}|x-y| \\
&= |x-y|,
\end{aligned}$$

so $F \in L$

(ii) No, it need not be true that $G \in M$. For example, let $g_n(x) = 2^n x$; each function g_n is continuous and satisfies $g_n(0) = 0$, so $g_n \in M$. However, $\sum_{n=1}^{\infty} 2^{-n} g_n(x)$ diverges for $x = 1$, so the function G is not even well-defines on $[0, 1]$.

5. Suppose that the operations $+$ and $\cdot$ can be extended to $\mathbb{R} \cup \{p\}$ to make $\mathbb{R} \cup \{p\}$ a field. Let $(-p)$ be the additive inverse of p, i.e. $p + (-p) = 0$. Since 0 is the unique element x satisfying $x + x = 0$, we must have $(-p) \neq p$. Thus $(-p) = r$ for some $r \in \mathbb{R}$. This implies $-r = p$, but if $r \in \mathbb{R}$ then $-r \in \mathbb{R}$, so we have $p \in \mathbb{R}$. This is a contradiction, since by assumption $p \notin \mathbb{R}$.

6. Since $U \subset \mathbb{R}^n$ is open, for each point $p \in U$ there is a number $r_p > 0$ so that $N_{r_p}(p) \subset U$. Write $p = (p_1, p_2, \ldots, p_n)$. For each $i = 1, \ldots, n$, let $q_i \in \mathbb{Q}$ with $|p_i - q_i| < r_p/(100n)$. Define $q_p = (q_1, \ldots, q_n)$. We have $q_p \in \mathbb{Q}^n$, and $N_{r_p/2}(q_p) \subset N_{r_p}(p) \subset U$. Finally, let r'_p be a rational number with $0 < r'_p < r/2$. Then $N_{r_{p'}}(q_p) \subset U$, and $p \in N_{r_{p'}}(q_p)$. This means that $U = \bigcup_{p \in U} N_{r_{p'}}(q_p)$. But observe that this is actually a countable union of open neighborhoods; each open neighborhood is of the form $N_t(q)$, where $t \in \mathbb{Q}$ and $q \in \mathbb{Q}^n$, and $\mathbb{Q} \times \mathbb{Q}^n$ is countable (indeed, we proved in lecture that $\mathbb{Q}$ is countable, and a finite Cartesian product of countable sets is countable).

7. Let $E = \mathbb{Q} \subset \mathbb{R}$. E is F_σ, since $\mathcal{Q}$ is countable and each singleton $\{q\}$ is a closed set. We can see that E is not open, since $0 \in E$ is not an interior point; indeed, for each $r > 0$, $N_r(0) \cap (\mathbb{R} \backslash \mathbb{Q})$ is non-empty. We can also see that E is not closed, because $\sqrt{2} \notin E$, but $\sqrt{2}$ is a limit point of E–we proved in lecture that $\sqrt{2}$ is the least upper bound for the set $\{x \in \mathbb{Q} \colon x \geq 0,\ x^2 < 2\} \subset E$. This implies that $\sqrt{2}$ is a limit point of E, and thus E does not contain all of its limit points.

8. (a) *Hint:* Aiming for a contradiction, let us suppose that there exists $f : [0,1] \to [0,1] \times [0,1]$ that is continuous, one-to-one and onto. Then the inverse function $g = f^{-1} : [0,1] \times [0,1] \to [0,1]$ is well-defined and continuous (prove this). Suppose that (x_0, y_0) is the unique point in $[0,1] \times [0,1]$ such that $g(x_0, y_0) = 1/2$. Then g maps $A = [0,1] \times [0,1] \setminus \{(x_0, y_0)\}$ bijectively onto $B = [0, 1/2) \cup (1/2, 1]$. However, A is connected (why?) and B is not (why?). This contradicts the theorem that the continuous image of any connected set is connected.

   Note: The same proof suitably modified shows that there cannot exist a continuous bijection between an interval and a rectangle.

   (b) *Hint:* Let us denote the image of the function f by

$$\mathrm{Range}(f) = \big\{f(x) : x \in M\big\}.$$

Since Range(f) is the continuous image of a compact set, it is compact and therefore closed. If, contrary to our desired conclusion, it is dense somewhere, that means it has nonempty interior and must therefore contain a closed rectangle R. Define $g = f^{-1}$ on R. Argue that $g(R)$ is an interval, and use part (a) to obtain the contradiction.

9. The statement is true. *Hint:* If f is not strictly monotone, first find an interval $I = (a - \epsilon, a + \epsilon)$ in $\mathbb{R}$ such that f is nonconstant on I, and attains its (local) maximum on I at $x = a$. Show that $f(I) = [f(a), b)$ for some $b \in \mathbb{R}$, contradicting the assumption that f is an open map.

10. *Fact:* Given $x \in M$, consider the set $\{x_n = f^{(n)}(x)\} \subseteq M$. Then there exists a subsequence $n_k \nearrow \infty$ such that $x_{n_k} \to x$.

 Proof. We recall that M is compact, hence every infinite subsequence in M converges. and after passing to a subsequence if necessary, we find that $x_{m_k} \to x_0$ as $k \to \infty$. Set $n_1 = m_1$, $n_2 = m_2 - m_1, \cdots, n_k = m_k - m_{k-1}$.

$$d(x, x_{n_k}) \leq d(x_{m_{k-1}}, x_{m_k}) \leq d(x_{m_{k-1}}, x_0) + d(x_{m_k}, x_0) \to \infty,$$

 as claimed. □

 (a) *Hint:* Fix $x, y \in M$. Use the fact above and its proof to find a single subsequence n_k such that $x_{n_k} \to x$, $y_{n_k} \to y$. Then,

$$d(f(x), f(y)) = d(x_1, y_1) \leq d(x_{n_k}, y_{n_k}) \to d(x, y) \text{ as } k \to \infty.$$

 This shows that $d(f(x), f(y)) \leq d(x, y)$. Combined with the hypothesis of the problem, this yields $d(f(x), f(y)) = d(x, y)$.

 (b) *Hint:* Suppose if possible that f is not onto, i.e., there exists $x \notin f(M)$. Use the above argument to show that x must be a limit point of $f(M)$. However $f(M)$ is compact, hence closed, so $x \in \overline{f(M)} = f(M)$, a contradiction.

11. The implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) and (v) $\implies$ (i) are left as an exercise. We sketch the proof of (iv) $\implies$ (v). Start by observing that $T(0) = 0$. Use the continuity of T at $0 \in V$ to find $\delta > 0$ such that

$$||T(y)|| = ||T(y) - T(0)|| \leq 1 \text{ whenever } ||y|| \leq \delta.$$

 Now for any $x \in V$, set $y = \delta x / ||x||$ and verify the desired conclusion with $C = 1/\delta$.

12. *Hint:* Use Cauchy-Schwarz to show that $||L|| = ||y||$.

13. *Hint:* Verify that $||I|| = b - a$, by first showing that $||I|| \leq b - a$, and the considering the "test function" $f(x) \equiv 1$.

14. The statement is true. emHint: Suppose that $\dim(V) = n$, and that $\{e_1, \cdots, e_n\}$ is a basis for V. Without loss of generality assume that

$$||x||_a = \sum_{i=1}^{n} |\alpha_i| \quad \text{where } x = \sum_{i=1}^{n} \alpha_i e_i.$$

 Show that $||x||_b \leq C||x||_a$. Use problem 11 to deduce that the identity operator is continuous from $(V, ||\cdot||_a)$ onto $(V, ||\cdot||_b)$. Consider the minimum value of this operator on the compact set $\{||x||_b = 1\}$.

15. Consider the space ℓ^1 of absolutely summable real sequences:

$$\ell^1 = \{\mathbf{a} = (a_1, a_2, \cdots) : \sum_n |a_n| < \infty\}.$$

The two norms

$$||\mathbf{a}||_1 = \sum_n |a_n| \quad \text{and} \quad ||\mathbf{a}||_2 = \left[\sum_n |a_n|^2\right]^{\frac{1}{2}}$$

are not equivalent. Verify that the sequence

$$\mathbf{a}_n = (1, \frac{1}{2}, \cdots, \frac{1}{n}, 0, 0, \cdots)$$

is Cauchy with respect to $|| \cdot ||_2$ but not with respect to $|| \cdot ||_1$.