Math 320 Midterm 2 Practice Problems

Instructions

(i) Midterm solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) Self-contained solutions are optimal. If in doubt whether to include the proof of a statement, ask your instructor.

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1. Let (M, d) be a metric space. For each part of this problem, identify a metric space M, a distance function d and a set $E \subseteq M$ that obeys the specified criteria, or show that no such set exists.

   (a) A closed subset that is not compact

   (b) A compact set that is not closed

   (c) An infinite compact set with no limit points in M

   (d) A connected set whose interior is disconnected

   (e) A connected set whose closure is disconnected

   (f) A complete set that is not compact

   (g) A compact set that is not complete

2. Let us recall that a subset D of a metric space is said to be *dense* in M if $\overline{D} = M$. A metric space (M, d) is said to be *separable* if it has a countable dense subset.

   (a) Prove that every totally bounded metric space is separable.

   (b) Give an example of a separable metric space that is not totally bounded.

3. Consider the metric space (ℓ^1, d), where

$$\ell^1 = \{\{x_n\} \colon \sup\{\sum_{i=1}^{k} |x_i| \colon k \in \mathbb{N}\} \text{ is finite}\},$$

   and

$$d(\{x_n\}, \{y_n\}) = \sup\{\sum_{i=1}^{k} |x_i - y_i| \colon k \in \mathbb{N}\}.$$

   (you can assume that d defines a metric; you do not have to prove this.)

   If you prefer, you can think of ℓ^1 as the set of sequences $\{x_n\}$ with $\sum_{i=1}^{\infty} |x_i|$ finite, and $d(\{x_n\}, \{y_n\}) = \sum_{i=1}^{\infty} |x_i - y_i|$, though we haven't defined infinite sums yet, which is why the above definition was provided.

   (a) Is (ℓ^1, d) compact? Prove that your answer is correct.

(b) Let $\mathbf{0} \in \ell^1$ be the sequence of all 0s. Is $N_1(\mathbf{0})$ totally bounded? Prove that your answer is correct.

(c) Is (ℓ^1, d) separable? Prove that your answer is correct.

(d) Bonus: Is (ℓ^1, d) complete? Prove that your answer is correct.

4. Let (M, d) be compact. Suppose that

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$$

is a decreasing sequence of nonempty closed sets in M, and that $\cap_{n=1}^{\infty} F_n$ is contained in some open set G. Show that $F_n \subset G$ for all but finitely many n.

Solution key

Disclaimer

(i) Some of the following discussion is intended to provide pointers for the solutions only. Flesh out these ideas in greater detail to arrive at a complete solution.

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1. (a) The set $E = [0, \infty)$ is closed but not compact in the metric space $M = \mathbb{R}$, equipped with the standard metric $d(x, y) = |x - y|$. To verify that E is closed, we observe that $E^c = (-\infty, 0)$ is open. However, E is not compact, since the open cover of E given by the sets $\{G_j = (-1, j) : j \geq 1\}$ admits no finite subcover.

   (b) There is no such set. Every compact set is necessarily closed, as shown in Theorem 2.34 of the textbook.

   (c) There is no such set. We will prove this by contradiction. If possible, let $E \subseteq M$ be an infinite compact set with no limit point in M. This means that for every $x \in M$, there exists $r = r_x > 0$ such that

$$E \cap \big[B(x; r) \setminus \{x\}\big] = \emptyset, \quad \text{where} \quad B(x; r) := \{y \in M : d(x, y) < r\}. \tag{1}$$

   The collection of open balls $\{B(x; r_x) : x \in M\}$ is clearly an open cover of M, and hence of E. Since E is compact, we can find $x_1, \cdots, x_n \in M$ such that

$$E \subseteq \bigcup_{i=1}^{n} B(x_i; r_{x_i}).$$

   The condition (1) implies that each ball $B(x_1; r_{x_i})$ can contain at most one point of E, namely x_i. Thus the cardinality of E is at most n, contradicting our assumption that E is infinite.

   (d) In the metric space $M = \mathbb{R}^2$ equipped with the standard Euclidean metric, let us consider the set $E = \overline{B(0; 1)} \cup \overline{B(2; 1)}$. Then E is connected (why?). However,

$$\text{int}(E) = B(0; 1) \cup B(2; 1)$$

   is disconnected. This is because $\text{int}(E)$ is the union of two non-empty separated sets $B(0; 1)$ and $B(2; 1)$.

   (e) No such set exists; if E is connected, then so is $\overline{E}$. Let us prove this by contradiction. Suppose if possible that $\overline{E}$ is disconnected, so it can be written as an union of two nonempty separated sets A and B, namely,

$$\overline{E} = A \cup B, \quad \text{with} \tag{2}$$

$$\overline{A} \cap B = \emptyset, \ A \cap \overline{B} = \emptyset. \tag{3}$$

   Intersecting both sides of (2) with E, we find that

$$E = E \cap \overline{E} = (A \cap E) \cup (B \cap E). \tag{4}$$

   We also note that by (3),

$$\overline{(A \cap E)} \cap (B \cap E) \subseteq \overline{A} \cap B = \emptyset, \quad \text{and} \quad (A \cap E) \cap \overline{(B \cap E)} \subseteq A \cap \overline{B} = \emptyset.$$

   Since E is assumed to be connected, this implies that either $A \cap E$ or $B \cap E$ is empty, otherwise E would be the union of two nontrivial separated sets $A \cap E$ and $B \cap E$. Assume without loss of generality that $B \cap E = \emptyset$, so $A \cap E = E$. This means that $E \subseteq A$, hence $\overline{E} \subset \overline{A}$. In view of (2), this last inclusion means that $B \subseteq \overline{A}$, implying that $B = B \cap \overline{A} = \emptyset$ by (3), a contradiction.

(f) $M = E = \mathbb{R}^d$ is complete but not compact. Non-compactness is easy to see by the Heine-Borel theorem, since $\mathbb{R}^d$ is unbounded. Fill out the following steps to show that $\mathbb{R}^d$ is complete, i.e., every Cauchy sequence is convergent. *Hint:* Let $\{x_n : n \geq 1\}$ be a Cauchy sequence in $\mathbb{R}$.

- First show that every Cauchy sequence is bounded, i.e., there exists a constant $R > 0$ such that $|x_n| \leq R$ for all $n \geq 1$.
- Since $\overline{B(0;R)}$ is compact in $\mathbb{R}^d$ (Heine-Borel), use (c) to deduce that $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$, whose limit x lies in $\overline{B(0;R)}$.
- Show that if $\{x_n\}$ is Cauchy and admits a subsequence $\{x_{n_k}\}$ that converges to x, then $x_n \to x$.

(g) No such set exists; every compact set is complete. Suppose that $E \subseteq M$ is compact. Let $\{x_n\}$ be a Cauchy sequence in E. We will show that there exists $x \in E$ such that $x_n \to x$, via the following sequence of steps:

- Assume without loss of generality that $\{x_n\}$ has infinitely many distinct elements. Use part (c) to show that $\{x_n\}$ has a subsequence that converges to a limit $x \in E$.
- Recycle the proof of the third step in part (f) to show that if $\{x_n\}$ and $x_{n_k} \to x$, then $x_n \to x$.

2. (a) For each $n \geq 1$, the total boundedness of M ensures the existence of a finite set D_n with the property

$$M = \bigcup_{x \in D_n} B\left(x; \frac{1}{n}\right).$$

Set $D = \cup_{n=1}^{\infty} D_n$. We claim that D is a countable dense subset of M, so that we can conclude that M is separable.

Let us prove this. On one hand, D is a countable union of finite sets, hence countable. On the other hand, for every $x \in M \setminus D$, and given any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $1/n < \epsilon$, and $x_n \in D_n \subseteq D$ such that $x \in B(x_n; 1/n) \subseteq B(x; \epsilon)$. Thus every point in $M \setminus D$ is a limit point of D, hence $M = D \cup (M \setminus D) = \overline{D}$, proving that D is dense in M.

(b) Let M be a countable infinite set and let d be the discrete metric on M. Then since M is countable and $\overline{M} = M$, we have that M is separable (indeed, every countable metric space is separable!). However, M is not totally bounded: if we select $\epsilon = 1/2$, the M cannot be covered by a union of finitely many open neighborhoods of radius ϵ.

3. (a) No; in order to show that (ℓ^1, d) is not compact, it suffices to find a sequence $\{p_n\}$ of points in ℓ^1 that does not have a convergent subsequence. For each $n \in \mathbb{N}$, define p_n to be the sequence whose n–the element is equal to $1/2$, and all other elements are 0. We have that for each index n, if $p_n = \{x_i\}$ then

$$\sup\{\sum_{i=1}^{k} |x_i| : k \in \mathbb{N}\} = 1/2.$$

Furthermore, if $n \neq m$ then $d(p_n, p_m) = 1$, so certainly this sequence does not have a convergent subsequence.

(b) No; consider the sequence $\{p_n\}$ defined in part a (each of these elements is contained in $N_1(\mathbf{0})$, and let $\epsilon = 1/2$. Since $d(p_n, p_m) = 1$ whenever $n \neq m$, we have that any open neighborhood of the form $N_{1/2}(q)$ can contain at most one element from the infinite sequence $\{p_n\}$. Thus $\{p_n\}$ cannot be contained in a finite union of open neighborhoods of radius 1/2, so certainly $N_1(\mathbf{0})$ cannot be contained in a finite union of open neighborhoods of radius 1/2.

(c) Yes! For each $k \in \mathbb{N}$, define

$$X_k = \big\{\{p_n\} \colon p_n \in \mathbb{Q} \text{ for all } n,\ p_n = 0 \text{ for all } n > k\big\}.$$

We have that X_k is in bijective correspondence with $\mathbb{Q}^k$, so in particular X_k is countable. Define $X = \bigcup_{k=1}^{\infty} X_k$; we have that X is a countable union of countable sets, and is thus countable.

It remains to show that X is dense in ℓ^1. Let $\{x_n\} \in \ell^1$. We will show that for all $\epsilon > 0$, there exists an element $\{y_n\} \in X$ with $d(x, y) < \epsilon$. Fix a choice of $\epsilon > 0$. Select $k_0 \in \mathbb{N}$ so that

$$\sum_{i=1}^{k_0} |x_i| > \sup\{\sum_{i=1}^{k_0} |x_i| \colon k \in \mathbb{N}\} - \epsilon/2,$$

or equivalently,

$$\sup\{\sum_{i=k_0+1}^{k} |x_i| \colon k \in \mathbb{N}\} < \epsilon/2.$$

Next, for each $n = 1, \ldots, k_0$, select a number $y_n \in \mathbb{Q}$ with $|x_n - y_n| < \epsilon/(2k_0)$. For $n > k_0$ define $y_n = 0$. The sequence $\{y_n\}$ defined in this way is an element of X_{k_0}, and is thus an element of X. We have that for every $k \in \mathbb{N}$,

$$\begin{aligned}
\sum_{i=1}^{k} |x_i - y_i| &\leq \sum_{i=1}^{k_0} |x_i - y_i| + \sum_{i=k_0+1}^{k} |x_i - y_i| \\
&< \sum_{i=1}^{k_0} \epsilon/(2k_0) + \sum_{i=k_0+1}^{k} |x_i| \\
&\leq \epsilon/2 + \sup\{\sum_{i=k_0+1}^{k} |p_i| \colon k \in \mathbb{N}\} \\
&< \epsilon/2 + \epsilon/2 \\
&< \epsilon
\end{aligned}$$

(d) Yes! let $\{p_m\}$ be a Cauchy sequence in (ℓ^1, d). For each index m, we will write $p_m = \{x_n^m\}_{n=1}^{\infty}$. Since $\{p_m\}$ is Cauchy, we have that for each $n \in \mathbb{N}$, $\{x_n^m\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Define x_n to be the limit point of the Cauchy sequence $\{x_n^m\}_{m=1}^{\infty}$.

Now let $\epsilon > 0$. Select N sufficiently large so that $d(p_m, p_{m'}) \leq \epsilon/4$ whenever $m, m' \geq N$. As in part c, select k_0 sufficiently large so that

$$\sup\{\sum_{i=k_0+1}^{k} |x_i^N| \colon k \in \mathbb{N}\} \leq \epsilon/4.$$

This implies that for all $m \geq N$, we have

$$\sup\{\sum_{i=k_0+1}^{k} |x_i^m| \colon k \in \mathbb{N}\} \leq \epsilon/2.$$

k_1 sufficiently large so that

$$\sup\{\sum_{i=k_1+1}^{k} |x_i| \colon k \in \mathbb{N}\} \leq \epsilon/4.$$

Let $k_2 = \max(k_0, k_1)$. Since $|x_i^m - x_i| \le |x_i^m| + |x_i|$ for each index i, we have that for all $m \ge N$ and all $k \in \mathbb{N}$,

$$\sum_{i=k_2+1}^{k} |x_i^m - x_i| \colon k \in \mathbb{N} \le 3\epsilon/4.$$

Next, since for each index $i = 1, \ldots, k_2$, we have that $\{x_i^m\}_{m=1}^{\infty}$ converges to x_i, for each $i = 1, \ldots, k_2$, there is an index M_i so that $d(x_i^m, x_i) < \epsilon/(4k_2)$ for all $m \ge M_i$. Let $M = \max_{1 \le i \le k_2} M_i$. We have that for all $m \ge M$,

$$\sum_{i=1}^{k_2} |x_i^m - x_i| < \sum_{i=1}^{k_0} \epsilon/(4k_0) = \epsilon/4.$$

Let $L = \max(N, M)$. We have that for all $m \ge L$ and for all $k \in \mathbb{N}$,

$$\begin{aligned}\sum_{i=1}^{k} |x_i^m - x_i| &\le \sum_{i=1}^{k_2} |x_i^m - x_i| + \sum_{i=k_2+1}^{k} |x_i^m - x_i| \\ &< \epsilon/4 + 3\epsilon/4 \\ &= \epsilon.\end{aligned}$$

All that remains is to show that the sequence $\{x_n\}$ is an element of ℓ_1. But this follows immediately from the fact that

$$\begin{aligned}\sup\{\sum_{i=1}^{k} |x_i| \colon k \in \mathbb{N}\} &\le \sum_{i=1}^{k_1} |x_i| + \sup\{\sum_{i=k_1+1}^{k} |x_i| \colon k \in \mathbb{N}\} && (5) \\ &\le \sum_{i=1}^{k_1} |x_i| + \epsilon/4, && (6)\end{aligned}$$

which is clearly finite.

4. The decreasing property of F_n and the condition $\cap_n F_n \subseteq G$ imply that

$$M \subseteq \bigcup_{n=1}^{\infty} F_n^c \cup G. \qquad \text{(why?)}$$

Since G and F_n^c are open sets and M is compact, we can extract a finite subcover:

$$M \subseteq F_{n_1}^c \cup F_{n_2}^c \cup \cdots F_{n_k}^c \cup G, \qquad \text{where } n_1 < n_2 < \cdots < n_k.$$

Since the sets F_n^c increase with n, the above inclusion means

$$M \subseteq F_{n_k}^c \cup G.$$

In other words, $F_{n_k} \cap G^c = \emptyset$, or $F_{n_k} \subseteq G$. Since the sets F_n are decreasing, this means that $F_n \subseteq G$ for any $n \ge n_k$, which is the desired conclusion.