Math 320 Assignment 8
Due Wednesday, November 7 at start of class

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) If your assignment has more than one page, staple them together.

(iv) Do not forget to include your name and SID.

(v) **All the problems in this set collectively define the completion of a metric space. The first four problems form HW 8. Please submit these problems on Nov 7. Problems 5-8 are for midterm practice only and should not be turned in.**

In this problem set, we will carefully define the completion of a metric space (X, d).

1. Let (X, d_X) be a metric space. Let Y be the set of all Cauchy sequences in X. We define a relation $\sim$ on Y as follows: $\{p_n\} \sim \{q_n\}$ if the sequence $\{r_n\}$ defined by

$$r_n = \begin{cases} p_{n/2}, & n \text{ is even,} \\ q_{(n+1)/2}, & n \text{ is odd.} \end{cases}$$

is Cauchy.

Prove that $\sim$ is an equivalence relation.

2. Let X' be the set of all equivalence classes of Y under the equivalence relation $\sim$ defined in Question 1. I.e. for each $\{p_n\} \in Y$ define

$$[\{p_n\}] = \{\{q_n\} \in Y : \{p_n\} \sim \{q_n\}\},$$

and define

$$X' = \{[\{p_n\}] : \{p_n\} \in Y\}.$$

We will define a metric on X' as follows: If $[\{p_n\}], [\{q_n\}] \in X'$, define

$$d_{X'}([\{p_n\}], [\{q_n\}]) = \lim_{n\to\infty} d_X(p_n, q_n).$$

Prove that if $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in X, then $\lim_{n\to\infty} d_X(p_n, q_n)$ always exists. Next, prove that if $\{p'_n\} \sim \{p_n\}$ and $\{q'_n\} \sim \{q_n\}$, then

$$\lim_{n\to\infty} d_X(p_n, q_n) = \lim_{n\to\infty} d_X(p'_n, q'_n).$$

These two statements imply that the metric $d_{X'}$ is well-defined. The metric space $(X', d_{X'})$ is called the completion of (X, d).

3. Let $\{p_n^1\}_{n=1}^\infty$, $\{p_n^2\}_{n=1}^\infty$, $\{p_n^3\}_{n=1}^\infty, \ldots$ be Cauchy sequences in X (and thus elements of Y). Note that p_n^2 doesn't mean p_n squared; the superscript denotes a second index. Suppose that the sequence

$\{[\{p_n^m\}_{n=1}^\infty]\}_{m=1}^\infty$ is a Cauchy sequence in X'. Prove that the sequence $\{p_n^n\}_{n=1}^\infty$ is a Cauchy sequence in X.

Remark: Some of you asked about the wording of the problem above, which is ambiguous as stated. If one chooses an arbitrary representative of the equivalence class of Cauchy sequences, the result is not true. If one chooses the representative carefully, the statement can be salvaged in a way that is helpful for solving Question 4. We have included the reformulation below.

A reformulation of Question 3: Let $\{P^{(m)} : m \geq 1\}$ denote a Cauchy sequence in X'. In other words, each $P^{(m)}$ is an equivalence class of Cauchy sequences in X, with the equivalence relation as in Question 1.

Show that for each $m \geq 1$, there exists a representative of $P^{(m)}$, namely a Cauchy sequence $\{p_n^{(m)} : n \geq 1\}$ in X and a subsequence $m_1 < m_2 < \cdots < m_k < \cdots$ with the following properties:

- $\{p_k^{(m_k)} : k \geq 1\}$ is a Cauchy sequence in X, and
- if P denotes the equivalence class in X' that contains the above sequence, then $P^{(m_k)} \to P$ as $k \to \infty$.

4. Prove that the metric space $(X', d_{X'})$ is complete.

5. Consider the function $\varphi \colon X \to X'$ which sends the element $p \in X$ to the Cauchy sequence $\{p_n\}$, with $p_n = p$ for every value of n. Prove that for all $p, q \in X$,
$$d_X(p, q) = d_{X'}(\varphi(p), \varphi(q)).$$

6. Prove that for every $[\{p_n\}] \in X'$ and every $\varepsilon > 0$, there exists an element $q \in X$ with $d_{X'}(\varphi(q), [\{p_n\}]) < \varepsilon$.

7. Define $\varphi(X) = \{\varphi(p) \colon p \in X\}$. Prove that $\overline{\varphi(X)} = X'$.

8. Let (Z, d_Z) be a complete metric space, and let $f \colon X \to Z$ be a function satisfying $d_X(p, q) = d_Z(f(p), f(q))$ for all $p, q \in X$. Prove that there exists a function $\psi \colon X' \to Z$ so that $f(p) = \psi \circ \varphi(p)$. In words: every isometry from X to a complete metric space factors through (X', d'_X).