Math 320 Assignment 7
Due Wednesday, October 31 at start of class

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) If your assignment has more than one page, staple them together.

(iv) Do not forget to include your name and SID.

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1. A k-cell R is a subset of $\mathbb{R}^k$ of the form

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots [a_k, b_k], \quad \text{with } -\infty < a_j < b_j < \infty \text{ for all } 1 \leq j \leq k.$$

Theorem 2.40 of the textbook shows that every k-cell is compact. Fill in the following sketch to arrive at an alternative proof of this fact.

(a) Suppose first that $k = 1$. Let $\{G_\alpha : \alpha \in A\}$ be an open cover of the 1-cell $I = [a, b]$. For $a < x \leq b$, set $I_x := [a, x]$, and define the set

$$X := \Big\{ x \in (a, b] : I_x \text{ admits a finite subcover from } \{G_\alpha\} \Big\}.$$

Show that X is nonempty and that $\sup(X) = b$. Show that b is in fact the maximum element of X. Does this prove that I is compact?

(b) Now suppose that $k = 2$, $R = [a_1, b_1] \times [a_2, b_2]$, and $\{U_\alpha\}$ is an open cover of R. Define $R_x := [a_1, x] \times [a_2, b_2]$. Use part (a) of this problem to show that the set

$$\mathcal{X} := \Big\{ x \in (a_1, b_1] : R_x \text{ admits a finite subcover from } \{U_\alpha\} \Big\}$$

has b_1 as its maximum, proving that R is compact.

(c) Generalize the argument above to show that a k-cell is compact for any $k \in \mathbb{N}$.

2. Let (M, d) be a metric space. A set $A \subseteq M$ is said to be *totally bounded* if given any $\epsilon > 0$ there exist finitely many points $x_1, \cdots, x_n \in M$ such that

$$A \subseteq \bigcup_{i=1}^{n} N_\epsilon(x_i),$$

where $N_r(x)$ denotes the open ball in M with centre x and radius r,

$$N_r(x) := \big\{ y \in M : d(x, y) < r \big\}.$$

(a) Show that every totally bounded set is also bounded, but that the converse need not be true.

(b) Show that every compact set is totally bounded.

(c) Find a metric space M and a totally bounded subset of it that is non-compact.

Remark: The concept of total boundedness is intricately related with compactness. Later in the course we will find a generalization of the Heine-Borel theorem that works in all metric spaces, not just $\mathbb{R}^n$. Namely, a set E in (M, d) is compact if and only if it is *complete* and totally bounded. A set E is complete if all Cauchy sequences in E are convergent, with the limit in E.

3. If $\{r_n\}$ is a sequence in a metric space (M, d), we define its *set of subsequential limits* to be the set
$$\{x \in M: \text{ there exists a subsequence of } \{r_n\} \text{ that converges to } x\}.$$
   (a) Find a sequence $\{r_n\}$ of real numbers such that the set of its subsequential limits is $[0, 1]$. (Be sure to prove that your sequence has the desired property.)
   (b) Prove that there is no sequence $\{r_n\}$ of real numbers such that the set of its subsequential limits is $(0, 1)$.

4. Recall the construction of the Cantor middle-third set $\mathcal{C}$ from Problem 3, Assignment 4. Show that the only nonempty connected subsets of $\mathcal{C}$ are the singletons. Sets with this property are said to be *totally disconnected*.