Math 320 Assignment 4
Due Wednesday, October 3 at start of class

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) If your assignment has more than one page, staple them together.

(iv) Do not forget to include your name and SID.

---

1. Let A and B be sets. We say that the cardinality of A is at most the cardinality of B (denoted $|A| \leq |B|$) if there exists an injection $f\colon A \to B$.

   (a) Prove that if A is infinite and $|A| \leq |B|$, then B is infinite.

   (b) Prove that if A is uncountable and $|A| \leq |B|$, then B is uncountable.

2. Let A and B be sets, with A non-empty.

   (a) Prove that if $|A| \leq |B|$, then there exists a surjection $g\colon B \to A$.

   (b) (*Extra credit, worth 10%*) Prove that if A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then A and B have the same cardinality, i.e., there exists a bijection between A and B. *Hint. This problem is really hard. Try to create a bijection out of the two injections, by looking at "forward and backward chains" of images.*

3. Define $C_0 = [0,1]$; this is a union of $2^0 = 1$ closed intervals, each of length $3^0 = 1$. Define $C_1 = [0,1/3] \cup [2/3,1]$; this set contains $2^1 = 2$ intervals, each of length $3^{-1} = 1/3$; it is obtained by removing the middle third of each interval from C_0. Define $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$; this set contains $2^2 = 4$ intervals, each of length $3^{-2} = 1/9$; it is obtained by removing the middle third of each interval from C_1. For each $i = 3,4,\ldots$, define C_i to be the union of 2^i closed intervals, each of length 3^{-i}, obtained by removing the middle third of each of the intervals from C_{i-1}. Define $\mathcal{C} = \bigcap_{i=0}^{\infty} C_i$. Prove that $\mathcal{C}$ is uncountable.

4. Let p be a prime number. Define the function $v_p\colon \mathbb{Q} \to \mathbb{R}$ as follows. For each nonzero rational number $a/b \in \mathbb{Q}$ (here $a \in \mathbb{Z}$ and $b \in \mathbb{N}$), there is a unique number $k \in \mathbb{Z}$ so that $a/b = p^k(c/d)$, where neither c nor d are divisible by p. Define $v_p(a/b) = p^{-k}$ for $a \neq 0$, with the convention that $v_p(0) = 0$. Define the function $d\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by $d(x,y) = v_p(x-y)$.

   (a) Prove that d is a metric.

   (b) Let $r > 0$, let $x \in \mathbb{Q}$ and let $y \in N_r(x) = \{z \in \mathbb{Q}\colon d(x,z) < r\}$. Prove that $N_r(x) = N_r(y)$. *Remark.* this is rather strange—every point in the ball $N_r(x)$ is also the "center" of the ball!