Math 320 Assignment 3
Due Wednesday, September 26 at start of class

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) Please staple your pages together when you submit your assignment.

(iv) Do not forget to include your name and SID.

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1. Read section 1.22 of the textbook. Then answer the following questions.

   (a) Let $p \geq 2$ be a fixed integer and $0 < x < 1$, $x \in \mathbb{R}$. Show that there exists a sequence of integers $a_1, a_2, \cdots, a_n \in \{0, 1, \cdots, p-1\}$ with the following property. Given any $\epsilon > 0$, there exists a rational r of the form

   $$r = \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n}{p^n}$$

   such that $|x - r| < \epsilon$. The integers $a_1, a_2, \cdots$ are called the *digits of the expansion of x with respect to base p.*

   (b) Suppose that x has a finite length expansion in base p, i.e., an expansion for which $a_n = 0$ for all but finitely many n. Prove that x has precisely two base p expansions.

   (c) If the base p expansion of x is not of finite length, show that it is unique.

   (d) Using your knowledge of decimal expansions, define a *repeating* base p expansion. Also describe when a base p expansion should be called *eventually repeating.*

   (e) Characterize the real numbers $0 < x < 1$ that have repeating base p expansions, and those that have eventually repeating base p expansions.

   Addendum Sept 24, 2018. Some students have requested clarification on the phrasing of Problem 1(a). Here is an alternate phrasing:

   Let $p \geq 2$ be a fixed integer and $0 < x < 1$, $x \in \mathbb{R}$. Show that there exists a sequence of integers $a_1, a_2, \ldots$ in $\{0, 1, \cdots, p-1\}$ with the following property. For every $\epsilon > 0$, there exists a natural number N so that for every natural number $n \geq N$, if we define

   $$r = \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n}{p^n},$$

   then $|x - r| < \epsilon$. The integers $a_1, a_2, \cdots$ are called the *digits of the expansion of x with respect to base p.*

2. Let $z, w \in \mathbb{C}$ with $\bar{z}w \neq 1$. Prove that:

   (a)

   $$\left| \frac{z - w}{1 - \bar{z}w} \right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

(b)

$$\left|\frac{z-w}{1-\bar{z}w}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

Hint: You may find it useful to first prove that there is an $r \geq 0$ and $v \in \mathbb{C}$ with $|v| = 1$ such that $z = rv$, and use this to reduce to the case of z real and positive. For the first case, elementary calculus can be used to show that $(r-w)(r-\bar{w}) < (1-rw)(1-r\bar{w})$.

3. (a) Show that any infinite set has a countably infinite subset.

   (b) Show that a set A is infinite if and only if it contains a proper subset that is of the same cardinality as itself.