Take-home Final - Math 440/508, Fall 2014

Due Monday December 8 by 5:00 pm. Please leave your paper in the instructor's mailbox in Math 126 or slide it under her office door in Math 214. Electronic submissions are also accepted.

Instructions: Your work will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. Applying ideas of Carathéodory, Koebe gave a proof of the Riemann mapping theorem by constructing explicitly a sequence of functions that converges to the desired conformal map. The goal of this problem is to fill out the details of Koebe's proof.

   A *Koebe domain* is, by definition, a simply connected domain $K_0 \subseteq \mathbb{D}$ that is not all of $\mathbb{D}$ and which contains the origin. Starting with a Koebe domain, the strategy is to find an injective function f_0 such that $f_0(K_0) = K_1$ is a Koebe domain "larger" than K_0. Then one iterates this process, finally obtaining functions $F_n = f_n \circ \cdots \circ f_0 : K_0 \to \mathbb{D}$ such that $F_n(K_0) = K_{n+1}$ and $\lim_{n\to\infty} F_n = F$ is a conformal map K_0 onto $\mathbb{D}$.

   The *inner radius* of a region $K \subseteq \mathbb{D}$ that contains the origin is defined by

   $$r_K = \sup\{\rho \geq 0 : B(0;\rho) \subseteq K\}.$$

   Also, a holomorphic injection $f : K \to \mathbb{D}$ is said to be an *expansion* if $f(0) = 0$ and $|f(z)| > |z|$ for all $z \in K \setminus \{0\}$.

   (a) Prove that if f is an expansion, then $r_{f(K)} \geq r_K$ and $|f'(0)| > 1$. [*Hint:* Write $f(z) = zg(z)$ and use the maximum principle to prove that $|f'(0)| = |g(0) > 1$.]

   Suppose we begin with a Koebe domain K_0 and a sequence of expansions $\{f_n : n \geq 0\}$ so that for every $n \geq 0$, the domain $K_{n+1} = f_n(K_n)$ is also a Koebe domain. We then define holomorphic maps $F_n = f_n \circ \cdots \circ f_0$.

   (b) Prove that for each n, the function F_n is an expansion. Moreover, show that

   $$F_n'(0) = \prod_{k=0}^{n} f_k'(0),$$

   and conclude that $\lim_{n\to\infty} |f_n'(0)| = 1$. [*Hint:* Prove that the sequence $\{|F_n'(0)|\}$ has a limit by showing that it is bounded above and monotone increasing. Use the Schwarz lemma.]

(c) Argue that without loss of generality one may choose the expansions $\{f_n\}$ such that $f_n'(0) > 0$. Using this, show that if the sequence is *osculating*, that is, if $r_{K_n} \to 1$ as $n \to \infty$, then F_n converges uniformly on compact subsets of K_0 to a conformal map $F : K_0 \to \mathbb{D}$. [*Hint:* If $r_{F(K_0)} \geq 1$ then F is surjective.]

To construct the desired osculating sequence we shall use the automorphisms $\psi_\alpha(z) = (\alpha - z)/(1 - \overline{\alpha} z)$.

(d) Given a Koebe domain K, choose a point $\alpha \in \mathbb{D}$ on the boundary of K such that $|\alpha| = r_K$ and also choose $\beta \in \mathbb{D}$ such that $\beta^2 = \alpha$. Let S denote the square root map, so that $S(0) = 0$. Why is the function $S \circ \psi_\alpha$ well-defined and holomorphic on K? Prove that the function $f : K \to \mathbb{D}$ defined by $f(z) = \psi_\beta \circ S \circ \psi_\alpha$ is an expansion. Moreover, show that $|f'(0)| = (1 + r_K)/(2\sqrt{r_K})$. [*Hint:* Apply the Schwarz lemma to the inverse function, namely $\psi_\alpha \circ g \circ \psi_\beta$ where $g(z) = z^2$.]

(e) Use part (d) to construct the desired sequence and complete the proof of the Riemann mapping theorem.