9. $f(x,y,z) = xyz,\ g(x,y,z) = x^2 + 2y^2 + 3z^2 = 6 \quad \Rightarrow \quad \nabla f = \langle yz, xz, xy \rangle$, $\lambda \nabla g = \langle 2\lambda x, 4\lambda y, 6\lambda z \rangle$. If $\lambda = 0$ then at least one of the coordinates is 0, in which case $f(x,y,z) = 0$. (None of these ends up giving a maximum or minimum.) If $\lambda \neq 0$, then $\nabla f = \lambda \nabla g$ implies $\lambda = (yz)/(2x) = (xz)/(4y) = (xy)/(6z)$ or $x^2 = 2y^2$ and $z^2 = \frac{2}{3}y^2$. Thus $x^2 + 2y^2 + 3z^2 = 6$ implies $6y^2 = 6$ or $y = \pm 1$. Thus the possible remaining points are $\left(\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}}\right)$. The maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.

18. $f(x,y) = 2x^2 + 3y^2 - 4x - 5 \quad \Rightarrow \quad \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad x = 1, y = 0$. Thus $(1, 0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \quad \Rightarrow \quad \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$, so $6y = 2\lambda y \quad \Rightarrow \quad$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \quad \Rightarrow \quad x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \leq 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1,0) = -7$.

44. (a) By Theorem 15.6.15 [ET 14.6.15], the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as the gradient vector.

(b) It is a minimum when $\mathbf{u}$ is in the direction opposite to that of the gradient vector (that is, $\mathbf{u}$ is in the direction of $-\nabla f$), since $D_{\mathbf{u}} f = |\nabla f| \cos\theta$ (see the proof of Theorem 15.6.15 [ET 14.6.15]) has a minimum when $\theta = \pi$.

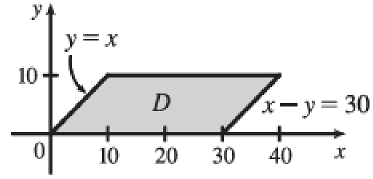
(c) The directional derivative is 0 when $\mathbf{u}$ is perpendicular to the gradient vector, since then $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}} f = |\nabla f| \cos\theta = \frac{1}{2} |\nabla f| \quad \Leftrightarrow \quad \cos\theta = \frac{1}{2} \quad \Leftrightarrow \quad \theta = \frac{\pi}{3}$.

32. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x,y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50}e^{-x}y & \text{if } x \geq 0, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X \geq Y$. Additionally, Yolanda will wait up to half an hour but no longer, so they won't meet unless $X - Y \leq 30$. Thus the probability that they meet is $P((X,Y) \in D)$ where D is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider D as a type II region, so



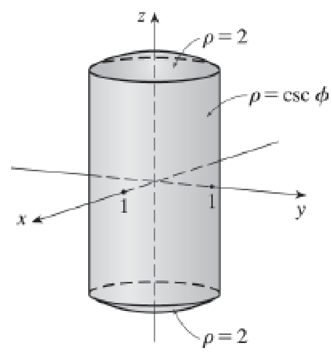
$$P((X,Y) \in D) = \iint_D f(x,y)\,dx\,dy = \int_0^{10}\int_y^{y+30} \tfrac{1}{50}e^{-x}y\,dx\,dy = \tfrac{1}{50}\int_0^{10} y\left[-e^{-x}\right]_{x=y}^{x=y+30}\,dy$$

$$= \tfrac{1}{50}\int_0^{10} y(-e^{-(y+30)} + e^{-y})\,dy = \tfrac{1}{50}(1 - e^{-30})\int_0^{10} ye^{-y}\,dy$$

By integration by parts (or Formula 96 in the Table of Integrals), this is

$\frac{1}{50}(1 - e^{-30})\left[-(y+1)e^{-y}\right]_0^{10} = \frac{1}{50}(1 - e^{-30})(1 - 11e^{-10}) \approx 0.020$. Thus there is only about a 2% chance they will meet. Such is student life!

14. $\rho \leq 2$ represents the solid sphere of radius 2 centered at the origin. Notice that $x^2 + y^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = \rho^2\sin^2\phi$. Then $\rho = \csc\phi \quad \Rightarrow \quad \rho\sin\phi = 1 \quad \Rightarrow \quad \rho^2\sin^2\phi = x^2 + y^2 = 1$, so $\rho \leq \csc\phi$ restricts the solid to that portion on or inside the circular cylinder $x^2 + y^2 = 1$.



36. Place the center of the sphere at $(0,0,0)$, let the diameter of intersection be along the z-axis, one of the planes be the xz-plane and the other be the plane whose angle with the xz-plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates the volume is given by

$V = \int_0^{\pi/6}\int_0^{\pi}\int_0^{a} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/6} d\theta\int_0^{\pi}\sin\phi\,d\phi\int_0^{a}\rho^2\,d\rho = \frac{\pi}{6}(2)\left(\frac{1}{3}a^3\right) = \frac{1}{9}\pi a^3$.

44. The given integral is equal to $\lim\limits_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho e^{-\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \lim\limits_{R\to\infty}\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{R}\rho^3e^{-\rho^2}\,d\rho\right)$.

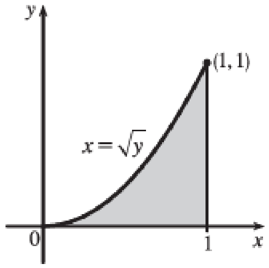
Now use integration by parts with $u = \rho^2$, $dv = \rho e^{-\rho^2}\,d\rho$ to get

$$\lim_{R\to\infty} 2\pi(2)\left(\rho^2\left(-\tfrac{1}{2}\right)e^{-\rho^2}\Big]_0^R - \int_0^R 2\rho\left(-\tfrac{1}{2}\right)e^{-\rho^2}\,d\rho\right) = \lim_{R\to\infty} 4\pi\left(-\tfrac{1}{2}R^2e^{-R^2} + \left[-\tfrac{1}{2}e^{-\rho^2}\right]_0^R\right)$$

$$= 4\pi\lim_{R\to\infty}\left[-\tfrac{1}{2}R^2e^{-R^2} - \tfrac{1}{2}e^{-R^2} + \tfrac{1}{2}\right] = 4\pi\left(\tfrac{1}{2}\right) = 2\pi$$

(Note that $R^2e^{-R^2} \to 0$ as $R \to \infty$ by l'Hospital's Rule.)

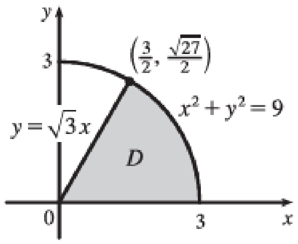
14.



$$\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\,dx\,dy = \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3}\,dy\,dx = \int_0^1 \frac{e^{x^2}}{x^3}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=x^2}\,dx$$

$$= \int_0^1 \tfrac{1}{2}xe^{x^2}\,dx = \tfrac{1}{4}e^{x^2}\Big]_0^1 = \tfrac{1}{4}(e-1)$$

21.



$$\iint_D \left(x^2+y^2\right)^{3/2} dA = \int_0^{\pi/3}\int_0^3 (r^2)^{3/2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/3} d\theta \int_0^3 r^4\,dr = \left[\theta\right]_0^{\pi/3}\left[\frac{1}{5}r^5\right]_0^3$$

$$= \frac{\pi}{3}\frac{3^5}{5} = \frac{81\pi}{5}$$