Homework 2 – Math 440/508

Due Friday October 3 at the end of lecture

Problem 1.

(a) Prove the following extension of the argument principle. Let D and f be as in the statement of the argument principle, and let g be analytic in $D \cup \partial D$. Then,

$$\frac{1}{2\pi i}\oint_{\partial D} g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^{m} g(a_i) - \sum_{j=1}^{n} g(p_j),$$

where $\{a_i : 1 \leq i \leq m\}$ and $\{p_j : 1 \leq j \leq n\}$ are respectively the zeros and poles of f in D counted with multiplicity.

(b) Let f be analytic on an open set containing $\overline{B(a;R)}$ and suppose that f is injective on $B(a;R)$. Set $\Omega = f[B(a;R)]$, $\gamma = \{a + Re^{i\theta} : 0 \leq \theta < 2\pi\}$, and use part (a) to deduce an integral formula for the inverse function of f; namely, for each $\xi \in \Omega$, $f^{-1}(\xi)$ is given by

$$f^{-1}(\xi) = \frac{1}{2\pi i}\oint_{\gamma} \frac{zf'(z)}{f(z)-\xi}\,dz.$$

Problem 2. Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : \text{Re } z > 0\}$. Show that the solution must be real. What happens to the solution as $\lambda \to 1$?

Problem 3. Answer the following questions. Provide explanations.

(a) How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $B(0;1)$?

(b) How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

(c) How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane? (*Hint:* Sketch the image of the imaginary axis and apply the argument principle to a large half disk.)

Problem 4. Let $f(z)$ be a continuously differentiable function on a domain D. Suppose that for all complex constants a and b, the increase in the argument of $f(z) + az + b$ around any small circle in D on which $f(z) + az + b \neq 0$ is non-negative. Show that $f(z)$ is analytic.

Problem 5.

(a) Verify that the statement of Rouché's theorem proved in class is a consequence of the following stronger version:

Let D be an open set in $\mathbb{C}$, γ a closed path in D, f, g two continuous complex-valued functions on $\gamma([0,1])$ such that

$$|f(\gamma(t)) - g(\gamma(t))| < |f(\gamma(t))| + |g(\gamma(t))| \quad \textit{for every } t \in [0,1].$$

Then $n(f \circ \gamma; 0) = n(g \circ \gamma; 0)$.

(b) The proof of the strong version of Rouché's theorem stated in part (a) rests on the following fact on winding numbers which we will revisit later in the course:

Let γ_0 and γ_1 be two closed paths in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with the same base point a. Then $n(\gamma_0; 0) = n(\gamma_1; 0)$ if and only if there exists a continuous map $H : [0,1] \times [0,1] \to \mathbb{C}^*$ such that

$$H(t,0) = \gamma_0(t), \quad H(t,1) = \gamma_1(t) \quad \textit{for } 0 \le t \le 1, \textit{ and}$$
$$H(0,s) = H(1,s) = a \quad \textit{for } 0 \le s \le 1.$$

One says that H is a homotopy that carries γ_0 to γ_1.

Assuming this fact for now, complete the proof of the strong version of Rouché's theorem.

(c) A man paces around a fountain two meters in diameter, his leashed dog with him. They start out from the same point. As the man walks, the dog wanders its own way, restrained only by the five foot leash. About twenty minutes after starting, the man and his dog find themselves at their common starting point. Show that the total number of turns around the fountain is the same for the dog and its master.

Remark added on October 1: The assumption of a common base point is a simplifying one, and not necessary. In light of the third version of Cauchy's theorem that we have proved in class this week, we now know that the result stated in part (b) holds for any two closed curves in $\mathbb{C}^*$ that are homotopic to each other, whether or not they have the same base point. In particular, this means that the conclusion of part (c) continues to hold even if the man and dog start out from different positions, as long as they return simultaneously to their respective starting points.

Problem 6. Let $f(z)$ be an analytic function on $B(0;1)$. Suppose there exists $0 < r < 1$ such that f restricted to $A(0;r,1)$ is injective. Show that f is injective on $B(0;1)$.

Problem 7. The goal of this problem is to deduce a version of the Cauchy integral formula involving winding numbers. Fill in the details of the steps outlined below: Let G be an open subset of the plane, $f : G \to \mathbb{C}$ an analytic function on G, and $\gamma : [0,1] \to \mathbb{C}$ a smooth closed curve in G such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$.

(a) Let $H = \{w \in \mathbb{C} : n(\gamma; w) = 0\}$. Define two auxiliary functions $\varphi : G \times G \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ as follows

$$\varphi(z,w) = \begin{Bmatrix} \dfrac{f(z)-f(w)}{z-w} \text{ if } z \neq w, \\ f'(z) \text{ if } z = w, \end{Bmatrix}$$

$$g(z) = \begin{Bmatrix} \oint_\gamma \varphi(z,w)dw \text{ if } z \in G, \\ \oint_\gamma \dfrac{f(w)}{w-z}dw \text{ if } z \in H. \end{Bmatrix}$$

Verify that g is well-defined, and an entire function.

(b) Argue that g is in fact the constant zero function.

(c) Combine the steps above to prove that for all $a \in G \setminus \gamma[0,1]$,

$$n(\gamma; a)f(a) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}dz.$$

Problem 8. Suppose that f is analytic on $\overline{B(0;3)}$, and never vanishes on $|z| = 3$. Assume that

$$\frac{1}{2\pi i}\oint_{|z|=3} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{|z|=3} z\frac{f'(z)}{f(z)}dz = 2, \quad \frac{1}{2\pi i}\oint_{|z|=3} z^2\frac{f'(z)}{f(z)}dz = -4.$$

Find the roots of f in $B(0;3)$.