Math 263 Assignment 8 Solutions

Problem 1. Given that $\mathbf{F}(x, y, z) = (2xz + y^2)\mathbf{i} + 2xy\mathbf{j} + (x^2 + 3z^2)\mathbf{k}$, find a function f such that $\mathbf{F} = \nabla f$ and use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve $C : x = t^2, y = t+1, z = 2t - 1$, $0 \leq t \leq 1$.

Solution. Set

$$f_x = 2xz + y^2, \tag{1}$$
$$f_y = 2xy, \text{ and} \tag{2}$$
$$f_z = x^2 + 3z^2. \tag{3}$$

Integrating the first equation with respect to x, we get

$$f(x, y, z) = x^2 z + xy^2 + g(y, z).$$

Therefore $f_y(x, y, z) = 2xy + g_y(y, z)$, so comparing with equation (2), we find that $g_y(y, z) = 0$. In other words, $g(y, z) = h(z)$. Thus $f(x, y, z) = x^2 z + xy^2 + h(z)$, from which we obtain $f_z(x, y, z) = x^2 + h(z)$. But $f_z(x, y, z) = x^2 + 3z^2$ from equation (3), so $h'(z) = 3z^2$, i.e., $h(z) = z^3 + K$. Hence one choice for f (setting $K = 0$) is $f(x, y, z) = x^2 z + xy^2 + z^3$.

In order to compute the line integral, note that $t = 0$ corresponds to the point $(0, 1, -1)$ and $t = 1$ corresponds to $(1, 2, 1)$, so by the fundamental theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2, 1) - f(0, 1, -1) = 6 - (-1) = 7.$$

□

Problem 2. Find the work done by the force field $\mathbf{F}(x, y) = e^{-y}\mathbf{i} - xe^{-y}\mathbf{j}$ in moving an object from $P(0, 1)$ to $Q(2, 0)$.

Solution. We first verify that the force field is conservative. Setting $\mathcal{P} = e^{-y}$ and $\mathcal{Q}Q = -xe^{-y}$, we see that

$$\frac{\partial \mathcal{P}}{\partial y} = -e^{-y} = \frac{\partial \mathcal{Q}}{\partial x}.$$

Thus there exists a function f such that $\mathbf{F} = \nabla f$, and the work done to move the particle from P to Q is independent of path.

In fact, we find such an f by setting $f_x = e^{-y}$ and $f_y = -xe^{-y}$. Integrating the first equationg gives $f(x, y) = xe^{-y} + g(y)$, from which we get $f_y = -xe^{-y} = g'(y)$.. Comparing with our earlier equation for f_y, we find that $g'(y) = 0$, so we can take $f(x, y) = xe^{-y}$ as a potential function for $\mathbf{F}$. Thus

$$W = \int \mathbf{F} \cdot d\mathbf{r} = f(2, 0) - f(0, 1) = 2 - 0 = 2.$$

□

Problem 3. Use Green's theorem to evaluate the line integral $\int_C \sin y dx + x \cos y dy$, where C is the ellipse $x^2 + xy + y^2 = 1$.

Solution. Let D denote the domain enclosed by the ellipse. By Green's theorem,

$$\int_C \sin y dx + x \cos y dy = \iint_D \left[\frac{\partial}{\partial x}(x \cos y) - \frac{\partial}{\partial y}(\sin y) \right] dA = \iint_D (\cos y - \cos y) dA = 0.$$

□

Problem 4. A particle starts at the point $(-2, 0)$, moves along the x-axis to (2,0), and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle$.

Solution. Let D denote the semicircular region bounded by C. By Green's theorem,

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_C x dx + (x^3 + 3xy^2) dy \\ &= \iint_D (3x^2 + 3y^2 - 0) dA \\ &= 3 \int_0^2 \int_0^\pi r^2 r d\theta dr \\ &= 12\pi. \end{aligned}$$

Note that we have coverted to polar coordinates at the second to last step. □

Problem 5. Find the curl and the divergence of the vector field $\mathbf{F}(x, y, z) = \langle e^x, e^{xy}, e^{xyz} \rangle$.

Solution. $\text{div}(\mathbf{F}) = \frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(e^{xy}) + \frac{\partial}{\partial z}(e^{xyz}) = e^x + xe^{xy} + xye^{xyz}$.

$$\begin{aligned} \text{curl}(\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & e^{xy} & e^{xyz} \end{vmatrix} \\ &= (xze^{xyz} - 0)\mathbf{i} - (yze^{xyz} - 0)\mathbf{j} + (ye^{xy} - 0)\mathbf{k} \\ &= \langle xze^{xyz}, -yze^{xyz}, ye^{xy} \rangle. \end{aligned}$$

□

Problem 6. Determine whether the force field $\mathbf{F}(x, y, z) = y \cos xy\mathbf{i} + x \cos xy\mathbf{j} - \sin z\mathbf{k}$ is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

Solution. We compute

$$\begin{aligned}
\operatorname{curl}(\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos xy & x\cos xy & -\sin z \end{vmatrix} \\
&= (0-0)\mathbf{i} - (0-0)\mathbf{j} + [(-xy\sin xy + \cos xy) - (-xy\sin xy + \cos xy)]\,\mathbf{k} = \mathbf{0}.
\end{aligned}$$

Since $\mathbf{F}$ is defines on all of $\mathbb{R}^3$ and the partial derivatives of the components functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then

$$f_x(x,y,z) = y\cos xy \implies f(x,y,z) = \sin xy + g(y,z) \implies f_y(x,y,z) = x\cos xy + g_y(y,z).$$

But $f_y(x,y,z) = x\cos xy$, so $g(y,z) = h(z)$, and $f(x,y,z) = \sin xy + h(z)$. Thus $f_z(x,y,z) = h'(z) = -\sin z$, so $h(z) = \cos z + K$; therefore a choice for f is $f(x,y,z) = \sin xy + \cos z + K$. □

Problem 7. Prove that $\operatorname{div}(\nabla f \times \nabla g) = 0$.

Solution. We will show that

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}. \tag{4}$$

The desired identity will follow from (4) by setting $\mathbf{F} = \nabla f$, $\mathbf{G} = \nabla g$, and recalling that $\operatorname{curl}\nabla f = \mathbf{0} = \operatorname{curl}\nabla g$.

To prove (4), we write $\mathbf{F} = P_1\mathbf{i} + Q_1\mathbf{j} + R_1\mathbf{k}$, and $\mathbf{G} = P_2\mathbf{i} + \mathbf{Q}_2\mathbf{j} + R_2\mathbf{k}$. Then

$$\begin{aligned}
\operatorname{div}(\mathbf{F} \times \mathbf{G}) &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{vmatrix} \\
&= \frac{\partial}{\partial x}\begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} - \frac{\partial}{\partial y}\begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix} + \frac{\partial}{\partial z}\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix} \\
&= \left[Q_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial Q_1}{\partial x} - Q_2\frac{\partial R_1}{\partial x} - R_1\frac{\partial Q_2}{\partial x}\right] - \left[P_1\frac{\partial R_2}{\partial y} + R_2\frac{\partial P_1}{\partial y} - P_2\frac{\partial R_1}{\partial y} - R_1\frac{\partial P_2}{\partial y}\right] \\
&\quad + \left[P_1\frac{\partial Q_2}{\partial z} + Q_2\frac{\partial P_1}{\partial z} - P_2\frac{\partial Q_1}{\partial z} - Q_1\frac{\partial P_2}{\partial z}\right] \\
&= \left[P_2\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right) + Q_2\left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right) + R_2\left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)\right] \\
&\quad - \left[P_1\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right) + Q_1\left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right) + R_1\left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\right] \\
&= \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}.
\end{aligned}$$

□