Math 263 Assignment 2 Solutions

1) Find the angle at which the following curves intersect:

$$\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}, \quad \mathbf{r}_2(t) = (1+t)\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}.$$

Solution. We first find the point of intersection of the two curves, by setting

$$\cos t_1 = 1 + t_2, \quad \sin t_1 = t_2^2, \quad t_1 = t_2^3, \tag{1}$$

and solving for t_1 and t_2. The first two equations may be combined to obtain $t_2^4 + (1+t_2)^2 = 1$, which upon simplification yields

$$\begin{aligned} (1+t_2^2)(1+t_2)(1-t_2) &= (1+t_2)^2, \\ \text{or, } (1+t_2)(1+t_2-(1-t_2)(1+t_2^2)) &= 0, \\ \text{or, } (1+t_2)(2t_2 - t_2^2 + t_2^3) &= 0, \\ \text{or, } t_2(1+t_2)(t_2^2 - t_2 + 2) &= 0. \end{aligned}$$

The two real solutions of the above equation are $t_2 = 0, -1$. Of these, $t_2 = -1$ is inadmissible since this would imply $t_1 = -1$, and the pair $t_1 = t_2 = -1$ does not satisfy the first two of the displayed equations in (1). Thus there is only one point of intersection, given by $t_1 = t_2 = 0$.

The tangent vectors to the curves $\mathbf{r}_1$ and $\mathbf{r}_2$ at the point of intersection are given by

$$\begin{aligned} \mathbf{r}_1'(0) &= -\sin 0\mathbf{i} + \cos 0\mathbf{j} + \mathbf{k} = \mathbf{j} + \mathbf{k}, \\ \mathbf{r}_2'(0) &= \mathbf{i} + 2(0)\mathbf{j} + 3(0)^2\mathbf{k} = \mathbf{i} \end{aligned}$$

respectively. The angle θ between these two vectors can therefore be computed using the dot product:

$$\cos\theta = \frac{\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(0)}{|\mathbf{r}_1'(0)||\mathbf{r}_2'(0)|} = 0, \qquad \text{so} \qquad \theta = \frac{\pi}{2}.$$

□

2) Evaluate :

$$\int_0^1 \left[\frac{t^2}{1+t^2}\mathbf{i} + t\sin(\pi t)\mathbf{j}\right] dt.$$

Solution. We compute the two integrals separately:

$$\int_0^1 \frac{t^2}{1+t^2}\,dt = \int_0^1 \left[1 - \frac{1}{1+t^2}\right] dt = \big[t - \arctan(t)\big]_{t=0}^{t=1} = 1 - \frac{\pi}{4},$$

and

$$\int_0^1 t\sin(\pi t)\,dt = -\frac{t}{\pi}\cos(\pi t)\Big|_{t=0}^{t=1} + \frac{1}{\pi}\int_0^1 \cos(\pi t)\,dt = \frac{1}{\pi}.$$

Note that we have used integration by parts to evaluate the second integral. Thus the answer is $(1-\pi/4)\mathbf{i} + \frac{1}{\pi}\mathbf{j}$. □

3) Find the tangential and normal components of the acceleration vector of the particle whose position function is given by

$$t\mathbf{i} + t^2\mathbf{j} + 3t\mathbf{k}.$$

Solution. Let $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 3t\mathbf{k}$. Then $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3\mathbf{k}$, and $\mathbf{r}''(t) = 2\mathbf{j}$. The tangential and normal components of the acceleration vector, denoted by a_T and a_N respectively are given by

$$\begin{aligned} a_T &= \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{4t}{\sqrt{10+4t^2}}, \\ a_N &= \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{|-6\mathbf{i}+2\mathbf{k}|}{\sqrt{10+4t^2}} = \sqrt{\frac{40}{10+4t^2}}. \end{aligned}$$

□

4) A gun is fired with angle of elevation 30°. What is the muzzle speed if the maximum height of the shell is 500 m?

Solution. If v_0 denotes the muzzle speed, the parametric equations of the trajectory of the shell are

$$x = (v_0\cos 30^\circ)t, \qquad y = (v_0\sin 30^\circ)t - \frac{1}{2}gt^2.$$

The maximum height of the shell will be attained when $y'(t) = 0$, i.e., $\frac{1}{2}v_0 - gt = 0$, or $t = v_0/(2g)$. Setting this value of t into y, we obtain

$$\frac{v_0^2}{8g} = 500, \qquad \text{or} \qquad v_0 = 20\sqrt{10g} \text{ meters.}$$

□

5) Find an equation of the osculating plane of the curve

$$x = \cos 2t, \quad y = t, \quad z = \sin 3t$$

at the point $(1,\pi,0)$.

Solution. Let $\mathbf{r}(t) = \cos 2t\mathbf{i} + t\mathbf{j} + \sin 3t\mathbf{k}$, so $\mathbf{r}'(t) = -2\sin 2t\mathbf{i} + \mathbf{j} + 3\cos 3t\mathbf{k}$, and $\mathbf{r}''(t) = -4\cos 2t\mathbf{i} - 9\sin 3t\mathbf{k}$. The plane to be determined passes through the point $(1,\pi,0)$ and is perpendicular to the binormal to the curve at this point. We proceed therefore to compute the direction of the binormal, which is perpendicular to both the tangent and normal vectors.

The unit tangent vector $\mathbf{T}(t)$ points in the direction of $\mathbf{r}'(t)$ and the unit normal vector in the direction of $\mathbf{T}'(t)$. More precisely,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}, \text{ so that } \mathbf{T}'(t) = \frac{\mathbf{r}''(t)}{|\mathbf{r}'(t)|} + \frac{d}{dt}\left(\frac{1}{|\mathbf{r}'(t)|}\right)\mathbf{r}'(t),$$

so

$$\mathbf{r}'(t) \times \mathbf{T}'(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t)|}.$$

Thus the direction of the binormal is the same as that of $\mathbf{r}'(t) \times \mathbf{r}''(t)$. Evaluated at $t = \pi$, the cross product in the last line is $= (\mathbf{j} - 3\mathbf{k}) \times (-4\mathbf{i}) = 4(\mathbf{k} + 3\mathbf{j})$. The equation of the osculating plane is therefore

$$3(y - \pi) + (1)(z - 0) = 0 \quad \text{or} \quad 3y + z = 3\pi.$$

□

6) At what point on the curve $x = t^3, y = 3t, z = t^4$ is the normal plane parallel to $z = \frac{3}{4}(x + y) - 2$?

Solution. We need to find the point on the curve where the tangent vector $3t^2\mathbf{i} + 3\mathbf{j} + 4t^3\mathbf{k}$ is parallel to the vector $3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, i.e.,

$$\frac{3t^2}{3} = \frac{3}{3} = \frac{4t^3}{-4}, \quad \text{in other words} \quad t = -1.$$

The point in question is therefore $(-1, -3, 1)$. □