MATHEMATICS 263 December 2004 Final Exam Solutions

1) The temperature in the solid ellipsoid $x^2 + xz + \frac{3}{2}z^2 + 2(y-2)^2 \le 11$ is given by

$$T(x, y, z) = \sqrt{y+3}\, e^{2x-z}$$

(a) Find a line that is perpendicular to the surface of the ellipsoid and passes through the point $P = (1, 1, 2)$. Call this line L.
(b) Calculate the rate of temperature change per unit distance at P in the direction inward along L.
(c) Estimate the temperature of the solid 0.09 units from point P inward along L.

Solution. (a) Let $g(x, y, z)$ be the LHS of the inequality. Then, an (outward pointing) normal to the surface of the solid at $(1, 1, 2)$ is given by

$$\vec{\mathbf{N}} = \nabla g(1, 1, 2) = \langle 2x + z, 4(y-2), x + 3z\rangle\,|_{(1,1,2)} = \langle 4, -4, 7\rangle$$

Therefore, the normal line at this point can be written

$$\boxed{\vec{\mathbf{r}}(t) = \langle 1, 1, 2\rangle + t\,\langle 4, -4, 7\rangle\,, \quad -\infty < t < \infty}$$

(b) We want the directional derivative at P in the direction of the *inward*–pointing normal, that is in the direction

$$\hat{\mathbf{n}} = \frac{-\vec{\mathbf{N}}}{|-\vec{\mathbf{N}}|} = \frac{\langle -4, 4, -7\rangle}{\sqrt{(-4)^2 + 4^2 + (-7)^2}} = \left\langle -\tfrac{4}{9}, \tfrac{4}{9}, -\tfrac{7}{9}\right\rangle$$

Since we have

$$\nabla T(x, y, z) = \left\langle 2\sqrt{y+3}\, e^{2x-z}, \tfrac{1}{2}(y+3)^{-1/2} e^{2x-z}, -\sqrt{y+3}\, e^{2x-z}\right\rangle$$

we may calculate the desired rate of change

$$D_{\hat{\mathbf{n}}} T(1, 1, 2) = \hat{\mathbf{n}} \cdot \nabla T(1, 1, 2) = \left\langle -\tfrac{4}{9}, \tfrac{4}{9}, -\tfrac{7}{9}\right\rangle \cdot \left\langle 4, \tfrac{1}{4}, -2\right\rangle = \boxed{-\tfrac{1}{9}}$$

(c) The temperature at the new point P' may be linearly approximated by

$$T(P') = T(P) + 0.09 D_{\hat{\mathbf{n}}} T(P) = 2 + 0.09\left(-\tfrac{1}{9}\right) = 2 - 0.01 = \boxed{1.99}$$

2) The mass m of an object with kinetic energy E and speed v is $m = 2E/v^2$. If a body has a measured kinetic energy of 200 and a measured speed of 100, but the measurements could have an error of $\pm 1\%$, what is the approximate maximum percentage error in the calculated value of the mass?

Solution. First note that $\frac{\partial m}{\partial E} = \frac{2}{v^2}$ and $\frac{\partial m}{\partial v} = -4\frac{E}{v^3}$. Also, we know that $\Delta E = \pm(200)(.01) = \pm 2$ and $\Delta v = \pm(100)(.01) = \pm 1$. We need to compute Δm.

$$\begin{aligned}
\Delta m &= m(200 + \Delta E, 100 + \Delta v) - m(200, 100) \\
&\approx \nabla m(200, 100) \cdot \langle \Delta E, \Delta v\rangle \qquad \text{by linear approximation} \\
&= \tfrac{\partial m}{\partial E}(200, 100)\Delta E + \tfrac{\partial m}{\partial v}(200, 100)\Delta v \\
&= \tfrac{2}{100^2}(\pm 2) - 4\tfrac{200}{100^3}(\pm 1) \\
&= \tfrac{\pm 4}{100^2} + \tfrac{\mp 8}{100^2} \\
&= \tfrac{\pm 4 + \mp 8}{100^2} = \tfrac{\pm 12}{100^2} = \tfrac{\pm 12}{10{,}000}
\end{aligned}$$

Since $m \approx 2\frac{200}{100^2} = \frac{4}{100}$ and $\pm\frac{12}{10{,}000} = \frac{4}{100}\left(\pm\frac{3}{100}\right)$ the approximate error in the mass is $\boxed{\pm 3\%}$.

3) Find the minimum and maximum values of $x^2 + 2y^2 - x$ in the region $x^2 + y^2 \le 1$.

Solution. Write $f(x, y) = x^2 + 2y^2 - x$. If the minimum/maximum value of f is achieved in $x^2 + y^2 < 1$, it must be achieved at a critical point. The critical points are the solutions of

$$0 = f_x = 2x - 1 \qquad 0 = f_y = 4y$$

So the only critical point is $\left(\frac{1}{2}, 0\right)$, where f takes the value $-\frac{1}{4}$. The other possibility is that f takes its min/max value on $x^2+y^2 = 1$. The value of $f(x, y)$ at $(x, y) = \big(\cos\theta, \sin\theta\big)$ is $g(\theta) = \cos^2\theta + 2\sin^2\theta - \cos\theta$. So the minimum/maximum value of f on the boundary is the same as the minimum/maximum value of $g(\theta)$, which we determine by finding the critical points of $g(\theta)$.

$$0 = g'(\theta) = -2\sin\theta\cos\theta + 4\sin\theta\cos\theta + \sin\theta = \sin\theta\big(2\cos\theta + 1\big)$$

Hence the critical points are at

$$\sin\theta = 0 \iff y = 0 \iff (x, y) = (\pm 1, 0) \qquad \cos\theta = -\tfrac{1}{2} \iff x = -\tfrac{1}{2} \iff (x, y) = \big(-\tfrac{1}{2}, \pm\tfrac{\sqrt{3}}{2}\big)$$

From the table of all possible candidates, below, we see that the minimum $\boxed{-\frac{1}{4}}$ and the maximum is $\boxed{\frac{9}{4}}$.

(x, y)	$\left(\frac{1}{2}, 0\right)$	$(1, 0)$	$(-1, 0)$	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$f(x, y)$	$-\frac{1}{4}$	0	2	$\frac{9}{4}$	$\frac{9}{4}$

4) Convert to polar coordinates and evaluate:

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \left(k + 3\sqrt{x^2 + y^2}\right)\, dy\, dx.$$

Express your answer in terms of the constant k.

Solution. Name the domain of integration D. This lies inside the vertical strip $0 \le x \le 2$, where it runs up from $y = 0$ to the curve where

$$y^2 = 2x - x^2, \quad \text{i.e.,} \quad x^2 - 2x + y^2 = 0, \quad \text{i.e.,} \quad (x-1)^2 + y^2 = 1.$$

Hence D is a semicircle of radius 1. In polar coordinates,

$$x^2 + y^2 = 2x \Leftrightarrow r^2 = 2r\cos\theta \Leftrightarrow r = 2\cos\theta,$$

and the angles of interest obey $0 \le \theta \le \pi/2$, so

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} (k + 3r) r\, dr\, d\theta = \int_{\theta=0}^{\pi/2} \left[\tfrac{k}{2} r^2 + r^3\right]_{r=0}^{2\cos\theta} d\theta = \int_0^{\pi/2} \left[2k\cos^2\theta + 8\cos^3\theta\right] d\theta$$

From the formula sheet $\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{\pi}{4}$ and $\int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{2}{3}$ so that $\boxed{I = \frac{k\pi}{2} + \frac{16}{3}}$. Notice that the coefficient of k can be found using geometry: it's just the area of the semicircle D, which is $\pi/2$.

5) Let $\vec{\mathbf{F}}(x,y) = (\sin y + y\cos x)\hat{\boldsymbol{\imath}} + (\sin x + x\cos y)\hat{\boldsymbol{\jmath}}$.

(a) Determine whether or not $\vec{\mathbf{F}}$ is conservative. If it is, find a potential function for $\vec{\mathbf{F}}$.

(b) Calculate $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, where C is the piece of the parabola $y = \frac{2}{\pi}x^2$ from $A=(0,0)$ to $B = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution. (a) $\vec{\mathbf{F}}$ might be conservative if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. In this case,

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(\sin y + y\cos x) = \cos y + \cos x = \frac{\partial}{\partial x}(\sin x + x\cos y) = \frac{\partial F_2}{\partial x}$$

so we need to find a function $\varphi(x,y)$ such that $\nabla\varphi = \vec{\mathbf{F}}$. There are may ways to find such a function. We could, by inspection, guess that $\varphi(x,y) = x\sin y + y\sin x$ would work and then verify that $\nabla\varphi(x,y) = (\sin y + y\cos x)\hat{\boldsymbol{\imath}} + (x\cos y + \sin x)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{F}}$, as desired. Let's try a more mechanical way to find this function. To have $\frac{\partial\varphi}{\partial x} = F_1$ we need

$$\varphi(x,y) = \int F_1 dx = \int (\sin y + y\cos x)\ dx = x\sin y + y\sin x + C(y)$$

To also have $\frac{\partial\varphi}{\partial y} = F_2$ we need $x\cos y + \sin x + C'(y) = F_2 = \sin x + x\cos y$. Thus $C'(y) = 0$, and hence $C(y)$ is a constant. We are free to choose $C(y) = 0$ so that $\boxed{\varphi(x,y) = x\sin y + y\sin x}$.

(b) The specified work integral is $\phi(B) - \phi(A) = \boxed{\pi}$.

6) Let D be the solid that is bounded below by the plane $2x + 2y + z + 2 = 0$ and is bounded above by the paraboloid $z = 4 - (x+1)^2 - (y+1)^2$. Let the field $\vec{\mathbf{F}}$ be given by

$$\vec{\mathbf{F}}(x,y,z) = \frac{\langle y, 1, z\rangle}{\sqrt{x^2+y^2}}$$

(a) Parameterize the curve of intersection of the plane and paraboloid in terms of the polar coordinate θ.

(b) Let S_1 be the portion of the surface of D formed by the paraboloid. Parameterize S_1.

(c) Let J denote the flux through S_1 into the solid D. Express J as an iterated double integral using the parameterization of part (b). Evaluate the inner integral. Evaluation of the remaining outer integral is not required.

Solution. (a) The curve of intersection is given by the system

$$2x + 2y + z + 2 = 0 \qquad z = 4 - (x+1)^2 - (y+1)^2$$

Substituting the value for z given by the second equation into the first gives $2x+2y+4-(x+1)^2-(y+1)^2+2 = 0$ or $4 - x^2 - y^2 = 0$, the circle centered at the origin of radius 2. This yields the parameterization

$$x = 2\cos\theta \qquad y = 2\sin\theta \qquad z = -2 - 2x - 2y = -2 - 4\cos\theta - 4\sin\theta \qquad 0 \le \theta \le 2\pi$$

(b) From part (a), we know that the (x,y) coordinates of the points of S_1 will be contained within the disc $x^2 + y^2 \le 4$, so using cylindrical coordinates, we have

$$x = r\cos\theta \quad y = r\sin\theta \quad z = 4-(r\cos\theta+1)^2-(r\sin\theta+1)^2 = 2-r^2-2r\cos\theta-2r\sin\theta \quad 0 \le r \le 2,\ 0 \le \theta \le 2\pi$$

(c) For the parameterization

$$\vec{\mathbf{r}}(r,\theta) = \big\langle r\cos\theta, r\sin\theta, 2 - r^2 - 2r\cos\theta - 2r\sin\theta\big\rangle$$

we have

$$\frac{\partial \vec{\mathbf{r}}}{\partial r} \times \frac{\partial \vec{\mathbf{r}}}{\partial \theta} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & -2r - 2\cos\theta - 2\sin\theta \\ -r\sin\theta & r\cos\theta & 2r\sin\theta - 2r\cos\theta \end{vmatrix}$$

$$= \hat{\boldsymbol{\imath}} \begin{vmatrix} \sin\theta & -2r - 2\cos\theta - 2\sin\theta \\ r\cos\theta & 2r\sin\theta - 2r\cos\theta \end{vmatrix} - \hat{\boldsymbol{\jmath}} \begin{vmatrix} \cos\theta & -2r - 2\cos\theta - 2\sin\theta \\ -r\sin\theta & 2r\sin\theta - 2r\cos\theta \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= \big\langle 2r + 2r^2\cos\theta, 2r + 2r^2\sin\theta, r \big\rangle$$

As the third component is positive, this is an upward normal and so outward from the surface. Therefore, we want to use

$$\hat{\mathbf{n}} dS = -\left(\tfrac{\partial \vec{\mathbf{r}}}{\partial r} \times \tfrac{\partial \vec{\mathbf{r}}}{\partial \theta}\right) dr\, d\theta = -\big\langle 2r + 2r^2\cos\theta, 2r + 2r^2\sin\theta, r \big\rangle\, dr\, d\theta$$

to calculate our flux integral. Converting $\vec{\mathbf{F}}$ to cylindrical coordinated yields

$$\vec{\mathbf{F}} = \langle \sin\theta, 1/r, 2/r - r - 2\cos\theta - 2\sin\theta \rangle$$

and so our flux integral is

$$\begin{aligned} J = \iint_{S_1} \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}\, dS &= \int_0^2 dr \int_0^{2\pi} d\theta\ \Big[\big\langle \sin\theta, \tfrac{1}{r}, \tfrac{2}{r} - r - 2\cos\theta - 2\sin\theta \big\rangle \\ &\qquad\qquad \bullet \Big(-\big\langle 2r + 2r^2\cos\theta, 2r + 2r^2\sin\theta, r \big\rangle \Big)\Big] \\ &= \boxed{\int_0^2 dr \int_0^{2\pi} d\theta\ \big[-2r\sin\theta - 2r^2\sin\theta\cos\theta - 4 + r^2 + 2r\cos\theta\big]} \\ &= \int_0^2 dr \int_0^{2\pi} d\theta \big[-2r\sin\theta - r^2\sin 2\theta - 4 + r^2 + 2r\cos\theta\big] \\ &= \int_0^2 dr\ \big[2r\cos\theta + \tfrac{1}{2}r^2\cos 2\theta - 4\theta + r^2\theta + 2r\sin\theta\big]_{\theta=0}^{\theta=2\pi} \\ &= \boxed{\textstyle\int_0^2 dr\ \big[-8\pi + 2\pi r^2\big]} = -8\pi r + \tfrac{2}{3}\pi r^3\Big|_{r=0}^{r=2} = -16\pi + \tfrac{16}{3}\pi = \boxed{-\tfrac{32}{3}\pi} \end{aligned}$$

For the other order of integration

$$\begin{aligned} J &= \int_0^{2\pi} d\theta \int_0^2 dr\ \big[-2r\sin\theta - 2r^2\sin\theta\cos\theta - 4 + r^2 + 2r\cos\theta\big] \\ &= \int_0^{2\pi} d\theta\ \big[-4\sin\theta - \tfrac{16}{3}\sin\theta\cos\theta - 8 + \tfrac{8}{3} + 4\cos\theta\big] \\ &= \boxed{\textstyle\int_0^{2\pi} d\theta\ \big[-4\sin\theta - \tfrac{16}{3}\sin\theta\cos\theta - \tfrac{16}{3} + 4\cos\theta\big]} \end{aligned}$$

7) Let $\mathcal{R}$ denote the solid region defined by the simultaneous inequalities

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad 1 \leq x^2 + y^2 + z^2 \leq 4$$

Let $\mathcal{S}$ denote the surface of $\mathcal{R}$.

(a) Sketch $\mathcal{R}$ and $\mathcal{S}$.

(b) Evaluate the outward flux of the following vector field through $\mathcal{S}$:

$$\vec{\mathbf{F}}(x, y, z) = \left\langle x^5 + y\sin(z),\, y^5 + z\sin(x),\, 10x^2y^2z - x \right\rangle.$$

(c) Find the flux of $\vec{\mathbf{F}}$ downward through the bottom of $\mathcal{S}$, i.e., through the flat part of $\mathcal{S}$ that lies in the plane $z = 0$.

Solution. (a) $\mathcal{R}$ is one quarter of the solid between a sphere of radius 1 and a sphere of radius 2.

(b) Here

$$\nabla \cdot \vec{\mathbf{F}} = \left[5x^4\right] + \left[5y^4\right] + \left[10x^2y^2\right] = 5(x^4 + 2x^2y^2 + y^4) = 5\left(x^2 + y^2\right)^2.$$

By the Divergence Theorem, the desired flux is

$$J = \iint_{\mathcal{S}} \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}\, dS = \iiint_{\mathcal{R}} \nabla \cdot \vec{\mathbf{F}}\, dV = 5 \iiint_{\mathcal{R}} (x^2 + y^2)^2\, dV.$$

Spherical coordinates are convenient for the volume integral.

$$\begin{aligned} J = 5 \iiint_{\mathcal{R}} \left(x^2 + y^2\right)^2 dV \qquad &= 5 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=1}^{2} \left[\rho^2 \sin^2 \phi\right]^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ = 5 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \tfrac{1}{7}\left[\rho^7\right]_{\rho=1}^{2} \sin^5 \phi\, d\phi\, d\theta &= 5 \tfrac{127}{7} \left(\tfrac{\pi}{2}\right) \int_{0}^{\pi/2} \sin^5 \phi\, d\phi \end{aligned}$$

From the formula sheet $\int_0^{\pi/2} \sin^5 \theta\, d\theta = \frac{8}{15}$ so that $\boxed{J = \frac{4 \times 127}{7 \times 3}\pi}$.

(c) Write $\mathcal{S}_0$ for the bottom of $\mathcal{S}$. On $\mathcal{S}_0$ we have $z = 0$; the outward unit normal is $\hat{\mathbf{n}} = -\mathbf{k}$, so

$$\vec{\mathbf{F}} \cdot \hat{\mathbf{n}} = x.$$

Polar coordinates work well for this:

$$\begin{aligned} \iint_{\mathcal{S}_0} \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}\, dS = \iint_{\mathcal{S}_0} x\, dA &= \int_{\theta=0}^{\pi/2} \int_{r=1}^{2} \left[r\cos\theta\right] r\, dr\, d\theta \\ &= \left(\int_{\theta=0}^{\pi/2} \cos\theta\, d\theta\right) \left(\int_{r=1}^{2} r^2\, dr\right) = (1)\left(\tfrac{2^3 - 1^3}{3}\right) = \boxed{\tfrac{7}{3}} \end{aligned}$$

8) Let $\vec{\mathbf{F}}(x,y,z) = (e^{x^2} + y)\,\hat{\boldsymbol{\imath}} + \big(\sin(y^3) + xz\big)\,\hat{\boldsymbol{\jmath}} + z^2\,\hat{\mathbf{k}}$. Use Stokes's theorem to evaluate $\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\mathcal{C}$ is the curve $x^2 + y^2 = 10$, $x + y + z = 4$ with positive orientation (i.e. counter–clockwise) as viewed from high on the z–axis.

Solution. Let D denote the disk $x + y + z = 4$, $x^2 + y^2 \le 10$ and let $\hat{\mathbf{n}}$ denote the upward pointing unit normal to D. By Stokes' theorem

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_D \nabla \times \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}\,dS$$

For the specified vector field

$$\nabla \times \vec{\mathbf{F}} = -x\,\hat{\boldsymbol{\imath}} + (z-1)\,\hat{\mathbf{k}}$$

Viewing $x + y + z = 4$ as $z = f(x,y)$ with $f(x,y) = 4 - x - y$

$$\begin{aligned} \hat{\mathbf{n}}\,dS &= \big(-f_x\,\hat{\boldsymbol{\imath}} - f_y\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big)\,dxdy = (\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})\,dxdy \\ \nabla \times \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}\,dS &= (-x, 0, f(x,y) - 1) \cdot (1,1,1)\,dxdy = (-x + f(x,y) - 1)\,dxdy \\ &= (3 - 2x - y)\,dxdy \end{aligned}$$

Since x and y are odd functions,

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{x^2+y^2\le 10} (3 - 2x - y)\,dxdy = 3 \iint_{x^2+y^2\le 10} dxdy = 3(10\pi) = \boxed{30\pi}$$