# MATHEMATICS 200 April 2003 Final Exam Solutions

[20] 1) At time $t = 0$ a particle has position and velocity vectors $\mathbf{r}(0) = \langle -1, 0, 0\rangle$ and $\mathbf{v}(0) = \langle 0, -1, 1\rangle$. At time $t$, the particle has acceleration vector

$$\mathbf{a}(t) = \langle \cos t, \sin t, 0\rangle$$

a) Find the position of the particle after $t$ seconds.

b) Show that the velocity and acceleration of the particle are always perpendicular for every $t$.

c) Find the equation of the tangent line to the particle's path at $t = -\pi/2$.

d) True or False: None of the lines tangent to the path of the particle pass through $(0, 0, 0)$. Justify your answer.

**Solution.** a)

$$\mathbf{v}'(t) = \mathbf{a}(t) = \langle \cos t, \sin t, 0\rangle \implies \mathbf{v}(t) = \langle \sin t + c_1, -\cos t + c_2, c_3\rangle$$

for some constants $c_1$, $c_2$, $c_3$. To satisfy $\mathbf{v}(0) = \langle 0, -1, 1\rangle$, we need $c_1 = 0$, $c_2 = 0$ and $c_3 = 1$. So $\mathbf{v}(t) = \langle \sin t, -\cos t, 1\rangle$. Similarly,

$$\mathbf{r}'(t) = \mathbf{v}(t) = \langle \sin t, -\cos t, 1\rangle \implies \mathbf{r}(t) = \langle -\cos t + d_1, -\sin t + d_2, t + d_3\rangle$$

for some constants $d_1$, $d_2$, $d_3$. To satisfy $\mathbf{r}(0) = \langle -1, 0, 0\rangle$, we need $d_1 = 0$, $d_2 = 0$ and $d_3 = 0$. So $\boxed{\mathbf{r}(t) = \langle -\cos t, -\sin t, t\rangle}$.

b)

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = \langle \sin t, -\cos t, 1\rangle \cdot \langle \cos t, \sin t, 0\rangle = \sin t \cos t - \cos t \sin t + 1 \times 0 = 0$$

so $\mathbf{v}(t) \perp \mathbf{a}(t)$ for all $t$.

c) At $t = -\frac{\pi}{2}$ the particle is at $\mathbf{r}\big(-\frac{\pi}{2}\big) = \big\langle 0, 1, -\frac{\pi}{2}\big\rangle$ and has velocity $\mathbf{v}\big(-\frac{\pi}{2}\big) = \langle -1, 0, 1\rangle$. Here is a vector parametric equation for the tangent line.

$$\boxed{\mathbf{r}(u) = \big\langle 0, 1, -\tfrac{\pi}{2}\big\rangle + u\,\langle -1, 0, 1\rangle}$$

d) $\boxed{\text{True}}$. Look at the path followed by the particle from the top so that we only see $x$ and $y$ coordinates. The path we see (call this the projected path) is $x(t) = -\cos t$, $y(t) = -\sin t$, which is a circle of radius one centred on the origin. Any tangent line to any circle always remains outside the circle. So no tangent line to the projected path can pass through the $(0, 0)$. So no tangent line to the path followed by the particle can pass through the $z$–axis and, in particular, through $(0, 0, 0)$.

[20] 2) Suppose $f(x, y)$ is a differentiable function and we know

$$\nabla f(3, 6) = \langle 7, 8\rangle$$

Suppose also that

$$\nabla g(1, 2) = \langle -1, 4\rangle\,,$$

and

$$\nabla h(1, 2) = \langle -5, 10\rangle\,.$$

Assuming $g(1, 2) = 3$, $h(1, 2) = 6$, and $z(s, t) = f\big(g(s, t), h(s, t)\big)$, find

$$\nabla z(1, 2)$$

**Solution.** By the chain rule

$$\begin{aligned}
\tfrac{\partial z}{\partial s}(s,t) &= \tfrac{\partial}{\partial s} f\big(g(s,t), h(s,t)\big) = \tfrac{\partial f}{\partial x}\big(g(s,t), h(s,t)\big)\tfrac{\partial g}{\partial s}(s,t) + \tfrac{\partial f}{\partial y}\big(g(s,t), h(s,t)\big)\tfrac{\partial h}{\partial s}(s,t)\\
\tfrac{\partial z}{\partial t}(s,t) &= \tfrac{\partial}{\partial t} f\big(g(s,t), h(s,t)\big) = \tfrac{\partial f}{\partial x}\big(g(s,t), h(s,t)\big)\tfrac{\partial g}{\partial t}(s,t) + \tfrac{\partial f}{\partial y}\big(g(s,t), h(s,t)\big)\tfrac{\partial h}{\partial t}(s,t)
\end{aligned}$$

In particular

$$\begin{aligned}
\tfrac{\partial z}{\partial s}(1,2) &= \tfrac{\partial f}{\partial x}\big(g(1,2), h(1,2)\big)\tfrac{\partial g}{\partial s}(1,2) + \tfrac{\partial f}{\partial y}\big(g(1,2), h(1,2)\big)\tfrac{\partial h}{\partial s}(1,2) = 7 \times (-1) + 8 \times (-5) = -47\\
\tfrac{\partial z}{\partial t}(1,2) &= \tfrac{\partial f}{\partial x}\big(g(1,2), h(1,2)\big)\tfrac{\partial g}{\partial t}(1,2) + \tfrac{\partial f}{\partial y}\big(g(1,2), h(1,2)\big)\tfrac{\partial h}{\partial t}(1,2) = 7 \times 4 \quad + 8 \times 10 \quad = 108
\end{aligned}$$

Hence $\boxed{\nabla z(1, 2) = \langle -47, 108\rangle}$.

[25] 3) Let

$$f(x,y) = xy(x+2y-6)$$

a) Find every critical point of $f(x,y)$ and classify each one.

b) Let $D$ be the region in the plane between the hyperbola $xy = 4$ and the line $x + 2y - 6 = 0$. Find the maximum and minimum values of $f(x,y)$ on $D$.

**Solution.** a) We have

$$\begin{array}{lll} f(x,y) = xy(x+2y-6) & f_x(x,y) = 2xy + 2y^2 - 6y & f_{xx}(x,y) = 2y \\ & f_y(x,y) = x^2 + 4xy - 6x & f_{yy}(x,y) = 4x \\ & & f_{xy}(x,y) = 2x + 4y - 6 \end{array}$$

At a critical point

$$\begin{aligned} f_x(x,y) = f_y(x,y) = 0 &\iff 2y(x+y-3) = 0 \text{ and } x(x+4y-6) = 0 \\ &\iff \{y = 0 \text{ or } x + y = 3\} \text{ and } \{x = 0 \text{ or } x + 4y = 6\} \\ &\iff \{x = y = 0\} \text{ or } \{y = 0,\ x + 4y = 6\} \\ &\qquad \text{or } \{x + y = 3,\ x = 0\} \text{ or } \{x + y = 3,\ x + 4y = 6\} \\ &\iff (x,y) = (0,0) \text{ or } (6,0) \text{ or } (0,3) \text{ or } (2,1) \end{aligned}$$

Here is a table giving the classification of each of the four critical points.

| critical point | $f_{xx}f_{yy} - f_{xy}^2$ | $f_{xx}$ | type |
|---|---|---|---|
| $(0,0)$ | $0 \times 0 - (-6)^2 < 0$ | | saddle point |
| $(6,0)$ | $0 \times 24 - 6^2 < 0$ | | saddle point |
| $(0,3)$ | $6 \times 0 - 6^2 < 0$ | | saddle point |
| $(2,1)$ | $2 \times 8 - 2^2 > 0$ | 2 | local min |

b) The shaded region in the sketch below is $D$.

$y$ $xy = 4$

$x + 2y = 6$

$(2,2)$

$(4,1)$

$x$

None of the critical points are in $D$. So the max and min must occur at either $(2,2)$ or $(4,1)$ or on $xy = 4$, $2 < x < 4$ (in which case $F(x) = f\big(x, \frac{4}{x}\big) = 4\big(x + \frac{8}{x} - 6\big)$ obeys $F'(x) = 4 - \frac{32}{x^2} = 0 \iff x = \pm 2\sqrt{2}$) or on $x + 2y = 6$, $2 < x < 4$ (in which case $f(x,y)$ is identically zero). So the min and max must occur at one of

| $(x,y)$ | $f(x,y)$ |
|---|---|
| $(2,2)$ | $2 \times 2(2 + 2 \times 2 - 6) = 0$ |
| $(4,1)$ | $4 \times 1(4 + 2 \times 1 - 6) = 0$ |
| $(2\sqrt{2}, 2/\sqrt{2})$ | $4(2\sqrt{2} + 2\sqrt{2} - 6) < 0$ |

The $\boxed{\text{maximum value is } 0}$ and the $\boxed{\text{minimum value is } 4(4\sqrt{2} - 6) \approx -1.37}$.

[20] 4) Consider the unit sphere

$$S = \left\{ (x,y,z) \;\middle|\; x^2+y^2+z^2=1 \right\}$$

in $\mathbb{R}^3$. Assume that the temperature at a point $(x,y,z)$ of $S$ is

$$T(x,y,z) = 40xy^2z$$

Find the hottest and coldest temperatures on $S$.

**Solution.** Set

$$f(x,y,z,\lambda) = T(x,y,z) - \lambda(x^2+y^2+z^2-1) = 40xy^2z - \lambda(x^2+y^2+z^2-1)$$

Then

$$\begin{aligned} f_x &= 40y^2z - 2x\lambda &&= 0 \\ f_y &= 80xyz - 2y\lambda &&= 0 \\ f_z &= 40xy^2 - 2z\lambda &&= 0 \\ f_\lambda &= x^2+y^2+z^2-1 = 0 \end{aligned}$$

Multiplying the first equation by $x$, the second equation by $y/2$ and the third equation by $z$ gives

$$\begin{aligned} 40xy^2z - 2x^2\lambda &= 0 \\ 40xy^2z - y^2\lambda &= 0 \\ 40xy^2z - 2z^2\lambda &= 0 \end{aligned}$$

Hence we must have

$$2x^2\lambda = y^2\lambda = 2z^2\lambda$$

If $\lambda = 0$, then $40y^2z = 0,\ 80xyz = 0,\ 40xy^2 = 0$ which is possible only if at least one of $x, y, z$ is zero so that $T(x,y,z) = 0$. Now suppose that $\lambda \neq 0$. Then

$$\begin{aligned} 2x^2 = y^2 = 2z^2 &\implies 1 = x^2+y^2+z^2 = x^2+2x^2+x^2 = 4x^2 \implies x = \pm\tfrac{1}{2},\ y^2 = \tfrac{1}{2},\ z = \pm\tfrac{1}{2} \\ &\implies T = 40\big(\pm\tfrac{1}{2}\big)\tfrac{1}{2}\big(\pm\tfrac{1}{2}\big) = \pm 5 \text{ (The sign of } x \text{ and } z \text{ need not be the same.)} \end{aligned}$$

So $\boxed{\text{the hottest temperature is } +5}$ and $\boxed{\text{the coldest temperature is } -5}$.

[25] 5) Consider the surface $S$ given by $z = e^{x^2+y^2}$.

a) Compute the volume under $S$ and above the disk $x^2+y^2 \le 9$.

b) The volume under $S$ and above a certain region $R$ in the $xy$–plane is

$$\int_0^1 \left( \int_0^y e^{x^2+y^2}\, dx \right) dy + \int_1^2 \left( \int_0^{2-y} e^{x^2+y^2}\, dx \right) dy$$

Sketch $R$ and express the volume as a single iterated integral with the order of integration reversed. **Do not compute either integral in part (b).**

**Solution.** a) The question should also have specified that the disk is in the $xy$–plane. Then the

$$\begin{aligned} \text{volume} &= \iint_{x^2+y^2\le 9} e^{x^2+y^2}\, dxdy = \int_0^3 dr \int_0^{2\pi} d\theta\ re^{r^2} = 2\pi \int_0^3 dr\ re^{r^2} = \pi e^{r^2}\Big|_0^3 \\ &= \boxed{\pi\big(e^9 - 1\big) \approx 25,453} \end{aligned}$$

b) The two integrals have domains

$$\left\{ (x,y) \;\middle|\; 0 \le y \le 1,\ 0 \le x \le y \right\} \qquad \left\{ (x,y) \;\middle|\; 1 \le y \le 2,\ 0 \le x \le 2-y \right\}$$

The domain is sketched in the figure on the left below.

$y$ $\quad$ $y$

$2$ $\quad$ $2$

$x = 2 - y$ $\quad$ $y = 2 - x$

$1$ $\quad$ $1$

$x = y$ $\quad$ $y = x$

$x$ $\quad$ $x$

$1$ $\quad$ $1$

b) Using the figure of the right above

$$\boxed{\text{volume} = \int_0^1 dx \int_x^{2-x} dy\ e^{x^2+y^2}}$$

[20] 6) let $a$, $b$ and $c$ be positive numbers, and let $T$ be the triangle whose vertices are $(-a, 0)$, $(b, 0)$ and $(0, c)$.

a) Assuming that the density is constant on $T$, find the center of mass of $T$.

b) The medians of $T$ are the line segments which join a vertex of $T$ to the midpoint of the opposite side. It is a well known fact the three medians of any triangle meet at a point, which is known as the centroid of $T$. Show that the centroid of $T$ is its centre of mass.

**Solution.** a) The side of the triangle from $(-a, 0)$ to $(0, c)$ has equation $cx - ay = -ac$. The side of the triangle from $(b, 0)$ to $(0, c)$ has equation $cx + by = bc$. The triangle has area $A = \frac{1}{2}(a + b)c$. It has centre of mass $(\bar{x}, \bar{y})$ with

$$\begin{aligned}
\bar{x} = \tfrac{1}{A} \iint_T x\ dxdy &= \tfrac{1}{A}\Big( \int_{-a}^0 dx \int_0^{c+\frac{c}{a}x} dy\ x + \int_0^b dx \int_0^{c-\frac{c}{b}x} dy\ x \Big) \\
&= \tfrac{1}{A}\Big( \int_{-a}^0 dx\ x\big(c + \tfrac{c}{a}x\big) + \int_0^b dx\ x\big(c - \tfrac{c}{b}x\big) \Big) \\
&= \tfrac{1}{A}\Big( \big[\tfrac{1}{2}cx^2 + \tfrac{c}{3a}x^3\big]_{-a}^0 + \big[\tfrac{1}{2}cx^2 - \tfrac{c}{3b}x^3\big]_0^b \Big) \\
&= 2\frac{\frac{1}{2}c(b^2 - a^2) + \frac{c}{3}(a^2 - b^2)}{(a + b)c} = \boxed{\tfrac{1}{3}(b - a)} \\
\bar{y} = \tfrac{1}{A} \iint_T y\ dxdy &= \tfrac{1}{A}\Big( \int_{-a}^0 dx \int_0^{c+\frac{c}{a}x} dy\ y + \int_0^b dx \int_0^{c-\frac{c}{b}x} dy\ y \Big) \\
&= \tfrac{1}{A}\Big( \int_{-a}^0 dx\ \tfrac{1}{2}\big(c + \tfrac{c}{a}x\big)^2 + \int_0^b dx\ \tfrac{1}{2}\big(c - \tfrac{c}{b}x\big)^2 \Big) \\
&= \tfrac{1}{A}\Big( \tfrac{a}{6c}\big(c + \tfrac{c}{a}x\big)^3\Big|_{-a}^0 - \tfrac{b}{6c}\big(c - \tfrac{c}{b}x\big)^3\Big|_0^b \Big) \\
&= 2\frac{\frac{ac^2}{6} + \frac{bc^2}{6}}{(a + b)c} = \boxed{\tfrac{c}{3}}
\end{aligned}$$

b) The midpoint of the side opposite $(-a, 0)$ is $\frac{1}{2}\big[(b, 0) + (0, c)\big] = \frac{1}{2}(b, c)$. The vector from $(-a, 0)$ to $\frac{1}{2}(b, c)$ is $\frac{1}{2}(b, c) - (-a, 0) = \big(a + \frac{b}{2}, \frac{c}{2}\big)$. So the line joining these two points has vector parametric equation

$$\mathbf{r}(t) = (-a, 0) + t\big(a + \tfrac{1}{2}b, \tfrac{1}{2}c\big)$$

The point $(\bar{x}, \bar{y})$ lies on this line since

$$\mathbf{r}\big(\tfrac{2}{3}\big) = \big(\tfrac{1}{3}(b - a), \tfrac{c}{3}\big) = (\bar{x}, \bar{y})$$

$y$

$(0, c)$

$(b/2, c/2)$

$x$

$(-a, 0)$ $\quad$ $(b, 0)$

Similarly, the midpoint of the side opposite $(b, 0)$ is $\frac{1}{2}(-a, c)$. The line joining these two points has vector parametric equation

$$\mathbf{r}(t) = (b, 0) + t\big(-b - \tfrac{1}{2}a, \tfrac{1}{2}c\big)$$

The point $(\bar{x}, \bar{y})$ lies on this line too, since

$$\mathbf{r}\big(\tfrac{2}{3}\big) = \big(\tfrac{1}{3}(b-a), \tfrac{c}{3}\big) = (\bar{x}, \bar{y})$$

It is not really necessary to check that $(\bar{x}, \bar{y})$ lies on the third median, but I will anyway. The midpoint of the side opposite $(0, c)$ is $\frac{1}{2}(b-a, 0)$. The line joining these two points has vector parametric equation

$$\mathbf{r}(t) = (0, c) + t\big(\tfrac{b}{2} - \tfrac{a}{2}, -c\big)$$

The point $(\bar{x}, \bar{y})$ lies on this median too, since

$$\mathbf{r}\big(\tfrac{2}{3}\big) = \big(\tfrac{1}{3}(b-a), \tfrac{c}{3}\big) = (\bar{x}, \bar{y})$$

[20] 7) Do part (a) or part (b) but not both.

a) Let $B$ denote the region inside the sphere $x^2+y^2+z^2 = 4$ and above the cone $x^2+y^2 = z^2$. Compute the moment of inertia

$$\iiint_B z^2\, dV$$

b) Find the area of the cone $z^2 = x^2 + y^2$ between $z = 1$ and $z = 16$.

**Solution.** a) In spherical coordinates,

$$x = R\sin\phi\cos\theta \qquad y = R\sin\phi\sin\theta \qquad z = R\cos\phi$$

the sphere $x^2 + y^2 + z^2 = 4$ is $R^2 = 4$ or $R = 2$ and the cone $x^2 + y^2 = z^2$ is $R^2\sin^2\phi = R^2\cos^2\phi$ or $\tan\phi = \pm 1$ or $\phi = \frac{\pi}{4}, \frac{3\pi}{4}$. So

$$\begin{aligned}
\text{moment} &= \int_0^2 dR \int_0^{\pi/4} d\phi \int_0^{2\pi} d\theta\ R^2\sin\phi(R\cos\phi)^2 = 2\pi\int_0^2 dR\ R^4\int_0^{\pi/4} d\phi\ \sin\phi\cos^2\phi \\
&= 2\pi\big[\tfrac{R^5}{5}\big]_0^2\big[-\tfrac{1}{3}\cos^3\phi\big]_0^{\pi/4} = \boxed{\tfrac{64}{15}\pi\big(1-\tfrac{1}{2\sqrt{2}}\big) \approx 8.665}
\end{aligned}$$

b) On the upper half of the cone

$$z = f(x, y) = \sqrt{x^2+y^2} \qquad f_x(x, y) = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y(x, y) = \frac{y}{\sqrt{x^2+y^2}}$$

so that

$$dS = \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dxdy = \sqrt{1 + \tfrac{x^2}{x^2+y^2} + \tfrac{y^2}{x^2+y^2}}\, dxdy = \sqrt{2}\, dxdy$$

and

$$\begin{aligned}
\text{Area} &= \iint_{1\le x^2+y^2\le 16^2} \sqrt{2}\, dxdy \\
&= \sqrt{2}\Big[\text{area of } \big\{\ (x, y)\ \big|\ x^2+y^2 \le 16^2\ \big\} - \text{area of } \big\{\ (x, y)\ \big|\ x^2+y^2 \le 1\ \big\}\Big] \\
&= \sqrt{2}\big[\pi 16^2 - \pi 1^2\big] = \boxed{255\sqrt{2}\pi \approx 1132.9}
\end{aligned}$$

[0 or 5] **Bonus Problem** Consider the sphere given by

$$(x-1)^2 + (y-2)^2 + (z+1)^2 = 2$$

Suppose that you are at the point $(2, 2, 0)$ on $S$, and you plan to follow the shortest path on $S$ to $(2, 1, -1)$. Express your initial direction as a cross product.

**Solution.** Switch to a new coordinate system with

$$X = x - 1 \qquad Y = y - 2 \qquad Z = z + 1$$

In this new coordinate system, the sphere has equation $X^2 + Y^2 + Z^2 = 2$. So the sphere is centred at $(X, Y, Z) = (0, 0, 0)$ and has radius $\sqrt{2}$. In the new coordinate system, the initial point $(x, y, z) = (2, 2, 0)$ has $(X, Y, Z) = (1, 0, 1)$ and our final point $(x, y, z) = (2, 1, -1)$ has $(X, Y, Z) = (1, -1, 0)$. Call the initial point $P$ and the final point $Q$. The shortest path will follow the great circle from $P$ to $Q$. A great circle on a sphere is the intersection of the sphere with a plane that contains the centre of the sphere. So the shortest path will lie on the plane that contains the three points $(X, Y, Z) = (0, 0, 0)$, $(1, 0, 1)$, $(1, -1, 0)$. This plane is $X + Y - Z = 0$. (Observe that all three points do indeed obey $X + Y - Z = 0$.) Since the shortest path lies on this plane, our direction vector must always lie on this plane and hence must always be perpendicular to $(1, 1, -1)$, which is the normal vector to this plane. The shortest path also remains on the sphere, so our initial direction must also be perpendicular to $(1, 0, 1)$ which is the normal vector to the sphere at our initial point $(X, Y, Z) = (1, 0, 1)$. Since our initial direction[†] must be perpendicular to both $(1, 1, -1)$ and $(1, 0, 1)$, it must be one of $\pm(1, 1, -1) \times (1, 0, 1)$. To get from $(1, 0, 1)$ to $(1, -1, 0)$ by the shortest path, our $Z$ coordinate should decrease from 1 to 0. So the $Z$ coordinate of our initial direction should be negative. This is the case for $\boxed{(1, 1, -1) \times (1, 0, 1)}$.

$Z$

$Y$

$X$

[†] Note that the change of coordinates $X = x - 1$, $Y = y - 2$, $Z = z + 1$ has absolutely no effect on any velocity or direction vector. If our position at time $t$ is $(x(t), y(t), z(t))$ in the original coordinate system, then it is $(X(t), Y(t), Z(t)) = (x(t) - 1, y(t) - 2, z(t) + 1)$ in the new coordinate system. The velocity vectors in the two coordinate systems $(x'(t), y'(t), z'(t)) = (X'(t), Y'(t), Z'(t))$ are identical.