MATHEMATICS 200 December 2002 Final Exam Solutions

[11] 1) The position of a particle at time t (measured in seconds s) is given by

$$\mathbf{r}(t) = t\cos\left(\tfrac{\pi t}{2}\right)\hat{\imath} + t\sin\left(\tfrac{\pi t}{2}\right)\hat{\jmath} + t\hat{\mathbf{k}}$$

a) Show that the path of the particle lies on the cone $z^2 = x^2 + y^2$.

b) Find the velocity vector and the speed at time t.

c) Suppose that at time $t = 1$s the particle flies off the path on a line L in the direction tangent to the path. Find the equation of the line L.

d) How long does it take for the particle to hit the plane $x = -1$ after it started moving along the straight line L?

Solution. a) Since

$$x(t)^2 + y(t)^2 = t^2\cos^2\left(\tfrac{\pi t}{2}\right) + t^2\sin^2\left(\tfrac{\pi t}{2}\right) = t^2 \qquad \text{and} \qquad z(t)^2 = t^2$$

are the same, the path of the particle lies on the cone $z^2 = x^2 + y^2$.

b)

$$\begin{aligned}
\text{velocity} = \mathbf{r}'(t) &= \boxed{\left[\cos\left(\tfrac{\pi t}{2}\right) - \tfrac{\pi t}{2}\sin\left(\tfrac{\pi t}{2}\right)\right]\hat{\imath} + \left[\sin\left(\tfrac{\pi t}{2}\right) + \tfrac{\pi t}{2}\cos\left(\tfrac{\pi t}{2}\right)\right]\hat{\jmath} + \hat{\mathbf{k}}} \\
\text{speed} = |\mathbf{r}'(t)| &= \sqrt{\left[\cos\left(\tfrac{\pi t}{2}\right) - \tfrac{\pi t}{2}\sin\left(\tfrac{\pi t}{2}\right)\right]^2 + \left[\sin\left(\tfrac{\pi t}{2}\right) + \tfrac{\pi t}{2}\cos\left(\tfrac{\pi t}{2}\right)\right]^2 + 1^2} \\
&= \Big[\cos^2\left(\tfrac{\pi t}{2}\right) - 2\tfrac{\pi t}{2}\cos\left(\tfrac{\pi t}{2}\right)\sin\left(\tfrac{\pi t}{2}\right) + \left(\tfrac{\pi t}{2}\right)^2\sin^2\left(\tfrac{\pi t}{2}\right) \\
&\qquad + \sin^2\left(\tfrac{\pi t}{2}\right) + 2\tfrac{\pi t}{2}\cos\left(\tfrac{\pi t}{2}\right)\sin\left(\tfrac{\pi t}{2}\right) + \left(\tfrac{\pi t}{2}\right)^2\cos^2\left(\tfrac{\pi t}{2}\right) + 1\Big]^{1/2} \\
&= \boxed{\sqrt{2 + \tfrac{\pi^2 t^2}{4}}}
\end{aligned}$$

c) At $t = 1$, the particle is at $\mathbf{r}(1) = (0, 1, 1)$ and has velocity $\mathbf{r}'(1) = \left(-\frac{\pi}{2}, 1, 1\right)$. So for $t \geq 1$, the particle is at

$$\boxed{(x, y, z) = (0, 1, 1) + (t-1)\left(-\tfrac{\pi}{2}, 1, 1\right)}$$

This is also a vector parametric equation for the line.

d) The question did not specify the speed of the particle after it started moving along L. I will assume that its speed remained constant. Then the x–coordinate of the particle at time t (for $t \geq 1$) is $-\frac{\pi}{2}(t-1)$. This takes the value -1 when $t - 1 = \frac{2}{\pi}$. So the particle hits $x = -1$, $\boxed{\frac{2}{\pi}}$ seconds after it flew off the cone.

[15] 2 a) Let f be an arbitrary differentiable function defined on the entire real line. Show that the function w defined on the entire plane as

$$w(x, y) = e^{-y} f(x - y)$$

satisfies the partial differential equation:

$$w + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$$

b) The equations $x = u^3 - 3uv^2$, $y = 3u^2v - v^3$ and $z = u^2 - v^2$ define z as a function of x and y. Determine $\frac{\partial z}{\partial x}$ at the point $(u, v) = (2, 1)$ which corresponds to the point $(x, y) = (2, 11)$.

Solution. a) By the product and chain rules

$$w_x(x, y) = e^{-y} f'(x - y) \qquad w_y(x, y) = -e^{-y} f(x - y) - e^{-y} f'(x - y)$$

Hence

$$w + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = e^{-y} f(x-y) + e^{-y} f'(x-y) - e^{-y} f(x-y) - e^{-y} f'(x-y) = 0$$

as desired.
b) Applying $\frac{\partial}{\partial x}$ to both sides of $x = u(x,y)^3 - 3u(x,y)v(x,y)^2$ and to both sides of $y = 3u(x,y)^2 v(x,y) - v(x,y)^3$ gives

$$\begin{aligned} 1 &= 3u(x,y)^2 \tfrac{\partial u}{\partial x}(x,y) - 3\tfrac{\partial u}{\partial x}(x,y)v(x,y)^2 - 6u(x,y)v(x,y)\tfrac{\partial v}{\partial x}(x,y) \\ 0 &= 6u(x,y)\tfrac{\partial u}{\partial x}(x,y)v(x,y) + 3u(x,y)^2 \tfrac{\partial v}{\partial x}(x,y) - 3v(x,y)^2 \tfrac{\partial v}{\partial x}(x,y) \end{aligned}$$

Subbing in $x = 2$, $y = 11$, $u = 2$, $v = 1$ gives

$$\begin{aligned} 1 &= 12\tfrac{\partial u}{\partial x}(2,11) - 3\tfrac{\partial u}{\partial x}(2,11) - 12\tfrac{\partial v}{\partial x}(2,11) = 9\tfrac{\partial u}{\partial x}(2,11) - 12\tfrac{\partial v}{\partial x}(2,11) \\ 0 &= 12\tfrac{\partial u}{\partial x}(2,11) + 12\tfrac{\partial v}{\partial x}(2,11) - 3\tfrac{\partial v}{\partial x}(2,11) = 12\tfrac{\partial u}{\partial x}(2,11) + 9\tfrac{\partial v}{\partial x}(2,11) \end{aligned}$$

From the second equation $\frac{\partial v}{\partial x}(2,11) = -\frac{4}{3}\frac{\partial u}{\partial x}(2,11)$. Subbing into the first equation gives $1 = 25\frac{\partial u}{\partial x}(2,11)$ so that $\frac{\partial u}{\partial x}(2,11) = \frac{1}{25}$ and $\frac{\partial v}{\partial x}(2,11) = -\frac{4}{75}$. Hence

$$\begin{aligned} &\tfrac{\partial z}{\partial x}(x,y) = 2u(x,y)\tfrac{\partial u}{\partial x}(x,y) - 2v(x,y)\tfrac{\partial v}{\partial x}(x,y) \\ \implies &\tfrac{\partial z}{\partial x}(2,11) = 4\tfrac{\partial u}{\partial x}(2,11) - 2\tfrac{\partial v}{\partial x}(2,11) = 4\tfrac{1}{25} + 2\tfrac{4}{75} = \tfrac{20}{75} = \boxed{\tfrac{4}{15}} \end{aligned}$$

[12] 3) You are standing at a lone palm tree in the middle of the Exponential Desert. The height of the sand dunes around you is given in meters by

$$h(x,y) = 100e^{-(x^2+2y^2)}$$

where x represents the number of meters east of the palm tree (west if x is negative) and y represents the number of meters north of the palm tree (south if y is negative).

a) Suppose that you walk 3 meters east and 2 meters north. At your new location, $(3,2)$, in what direction is the sand dune sloping most steeply downward?

b) If you walk north from the location described in part (a), what is the instantaneous rate of change of height of the sand dune?

c) If you are standing at $(3,2)$ in what direction should you walk to ensure that you remain at the same height?

d) Find the equation of the curve through $(3,2)$ that you should move along in order that you are always pointing in a steepest descent direction at each point of this curve.

Solution. We have

$$\nabla h(x,y) = -200e^{-(x^2+2y^2)}(x, 2y) \text{ and, in particular, } \nabla h(3,2) = -200e^{-17}(3,4)$$

a) At $(3,2)$ the dune slopes downward the most steeply in the direction opposite $\nabla h(3,2)$, which is $\boxed{(3,4)}$.
b) The rate is $D_{\hat{\jmath}} h(3,2) = \nabla h(3,2) \cdot \hat{\boldsymbol{\jmath}} = \boxed{-800e^{-17}}$.
c) To remain at the same height, you should walk perpendicular to $\nabla h(3,2)$. So you should walk in one of the directions $\boxed{\pm\left(\frac{4}{5}, -\frac{3}{5}\right)}$.
d) Suppose that you are walking along a steepest descent curve. Then the direction from (x,y) to $(x+dx, y+dy)$, with (dx,dy) infinitesimal, must be opposite to $\nabla h(x,y) = -200e^{-(x^2+2y^2)}(x,2y)$. Thus (dx,dy) must be parallel to $(x,2y)$ so that the slope

$$\frac{dy}{dx} = \frac{2y}{x} \implies \frac{dy}{y} = 2\frac{dx}{x} \implies \ln y = 2\ln x + C$$

We must choose C to obey $\ln 2 = 2\ln 3 + C$ in order to pass through the point $(3, 2)$. Thus $C = \ln \frac{2}{9}$ and the curve is $\ln y = 2\ln x + \ln \frac{2}{9}$ or $\boxed{y = \frac{2}{9}x^2}$.

[12] 4) Find all the critical points of the function

$$f(x, y) = x^4 + y^4 - 4xy$$

defined in the xy–plane. Classify each critical point as a local minimum, maximum or saddle point. Explain your reasoning.

Solution. We have

$$\begin{aligned} f(x, y) = x^4 + y^4 - 4xy \quad & f_x(x, y) = 4x^3 - 4y \quad & f_{xx}(x, y) = 12x^2 \\ & f_y(x, y) = 4y^3 - 4x \quad & f_{yy}(x, y) = 12y^2 \\ & & f_{xy}(x, y) = -4 \end{aligned}$$

At a critical point

$$\begin{aligned} f_x(x, y) = f_y(x, y) = 0 &\iff y = x^3 \text{ and } x = y^3 \iff x = x^9 \text{ and } y = x^3 \iff x(x^8 - 1) = 0,\ y = x^3 \\ &\iff (x, y) = (0, 0) \text{ or } (1, 1) \text{ or } (-1, -1) \end{aligned}$$

Here is a table giving the classification of each of the three critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0, 0)$	$0 \times 0 - (-4)^2 < 0$		saddle point
$(1, 1)$	$12 \times 12 - (-4)^2 > 0$	12	local min
$(-1, -1)$	$12 \times 12 - (-4)^2 > 0$	12	local min

[12] 5 a) By finding the points of tangency determine the values of c for which $x + y + z = c$ is a tangent plane to the surface $4x^2 + 4y^2 + z^2 = 96$.

b) Use the method of Lagrange Multipliers to determine the absolute maximum and minimum values of the function $f(x, y, z) = x + y + z$ along the surface $g(x, y, z) = 4x^2 + 4y^2 + z^2 = 96$.

c) Why do you get the same answers in (a) and (b)?

Solution. a) A normal vector to $F(x, y, z) = 4x^2 + 4y^2 + z^2 = 96$ at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = (8x_0, 8y_0, 2z_0)$. (Note that this normal vector is never the zero vector because $(0, 0, 0)$ is not on the surface.) So the tangent plane to $4x^2 + 4y^2 + z^2 = 96$ at (x_0, y_0, z_0) is

$$8x_0(x - x_0) + 8y_0(y - y_0) + 2z_0(z - z_0) = 0 \qquad \text{or} \qquad 8x_0x + 8y_0y + 2z_0z = 8x_0^2 + 8y_0^2 + 2z_0^2$$

This plane is of the form $x + y + z = c$ if and only if $8x_0 = 8y_0 = 2z_0$. A point (x_0, y_0, z_0) with $8x_0 = 8y_0 = 2z_0$ is on the surface $4x^2 + 4y^2 + z^2 = 96$ if and only if

$$4x_0^2 + 4y_0^2 + z_0^2 = 4x_0^2 + 4x_0^2 + (4x_0)^2 = 96 \iff 24x_0^2 = 96 \iff x_0^2 = 4 \iff x_0 = \pm 2$$

When $x_0 = \pm 2$, we have $y_0 = \pm 2$ and $z_0 = \pm 8$ (upper signs go together and lower signs go together) so that the tangent plane $8x_0x + 8y_0y + 2z_0z = 8x_0^2 + 8y_0^2 + 2z_0^2$ is

$$\begin{aligned} 8(\pm 2)x + 8(\pm 2)y + 2(\pm 8)z = 8(\pm 2)^2 + 8(\pm 2)^2 + 2(\pm 8)^2 \qquad &\text{or} \qquad \pm x \pm y \pm z = 2 + 2 + 8 \\ &\text{or} \qquad x + y + z = \mp 12 \implies \boxed{c = \pm 12} \end{aligned}$$

b) Set

$$F(x, y, z, \lambda) = x + y + z - \lambda(4x^2 + 4y^2 + z^2 - 96)$$

Then

$$\begin{aligned} F_x &= 1 - 8x\lambda &&= 0 \\ F_y &= 1 - 8y\lambda &&= 0 \\ F_z &= 1 - 2z\lambda &&= 0 \\ F_\lambda &= 4x^2 + 4y^2 + z^2 - 96 &&= 0 \end{aligned}$$

The first three equations give

$$x = \tfrac{1}{8\lambda} \qquad y = \tfrac{1}{8\lambda} \qquad z = \tfrac{1}{2\lambda} \qquad \text{with } \lambda \neq 0$$

Subbing this into the fourth equation gives

$$4\big(\tfrac{1}{8\lambda}\big)^2 + 4\big(\tfrac{1}{8\lambda}\big)^2 + \big(\tfrac{1}{2\lambda}\big)^2 = 96 \iff \big(\tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{4}\big)\tfrac{1}{\lambda^2} = 96 \iff \lambda^2 = \tfrac{3}{8}\tfrac{1}{96} = \tfrac{1}{8\times 32} \iff \lambda = \pm\tfrac{1}{16}$$

Hence $x = \pm 2$, $y = \pm 2$ and $z = \pm 8$ so that the largest and smallest values of $x+y+z$ on $4x^2+4y^2+z^2-96$ are $\pm 2 \pm 2 \pm 8$ or $\boxed{\pm 12}$.

c) The level surfaces of $x + y + z$ are planes with equation of the form $x + y + z = c$. To find the largest (smallest) value of $x + y + z$ on $4x^2 + 4y^2 + z^2 = 96$ we keep increasing (decreasing) c until we get to the largest (smallest) value of c for which the plane $x + y + z = c$ intersects $4x^2 + 4y^2 + z^2 - 96$. For this value of c, $x + y + z = c$ is tangent to $4x^2 + 4y^2 + z^2 = 96$.

[8] 6) Evaluate the following integral:

$$\int_{-2}^{2}\int_{x^2}^{4} \cos\left(y^{3/2}\right)\, dy\, dx$$

Solution. The domain of integration is $-2 \le x \le 2$, $x^2 \le y \le 4$. This is sketched below. To exchange the order of integration, we reexpress the domain as $0 \le y \le 4$, $-\sqrt{y} \le x \le \sqrt{y}$.

y $\quad y = x^2$

$y = 4$

x

$x = -2$ $\quad x = 2$

Exchanging the order of integration

$$\begin{aligned} \int_{-2}^{2}\int_{x^2}^{4} \cos\left(y^{3/2}\right)\, dy\, dx &= \int_0^4 dy \int_{-\sqrt{y}}^{\sqrt{y}} dx\ \cos\left(y^{3/2}\right) \\ &= \int_0^4 dy\ 2\sqrt{y}\cos\left(y^{3/2}\right) \\ &= \tfrac{4}{3}\int_0^8 dt\ \cos t \quad \text{where } t = y^{3/2},\ dt = \tfrac{3}{2}\sqrt{y}\, dy \\ &= \tfrac{4}{3}\sin t\Big|_0^8 = \boxed{\tfrac{4}{3}\sin 8 \approx 1.319} \end{aligned}$$

[15] 7) Let D be the region in the xy–plane which is inside the circle $x^2 + (y-1)^2 = 1$ but outside the circle $x^2 + y^2 = 2$. Determine the mass of this region if the density is given by

$$\rho(x, y) = \frac{2}{\sqrt{x^2 + y^2}}$$

Solution. The domain is pictured below.

y

$x^2+(y-1)^2=1$

x

$x^2+y^2=2$

The two circles intersect when $x^2+y^2=2$ and

$$y^2-(y-1)^2=1 \iff 2y-1=1 \iff y=1 \text{ and } x=\pm 1$$

In polar coordinates $x^2+y^2=2$ is $r=\sqrt{2}$ and $x^2+(y-1)^2=x^2+y^2-2y+1=1$ is $r^2-2r\sin\theta=0$ or $r=2\sin\theta$. The two curves intersect when $r=\sqrt{2}$ and $\sqrt{2}=2\sin\theta$ so that $\theta=\frac{\pi}{4}$ or $\frac{3}{4}\pi$. So

$$\begin{aligned}\text{mass} &= \int_{\pi/4}^{3\pi/4} d\theta \int_{\sqrt{2}}^{2\sin\theta} dr\ r\tfrac{2}{r} = 2\int_{\pi/4}^{3\pi/4} d\theta\ \big[2\sin\theta-\sqrt{2}\big] = 4\int_{\pi/4}^{\pi/2} d\theta\ \big[2\sin\theta-\sqrt{2}\big] \\ &= 4\big[-2\cos\theta-\sqrt{2}\theta\big]_{\pi/4}^{\pi/2} = \boxed{4\sqrt{2}-\sqrt{2}\pi \approx 1.214}\end{aligned}$$

[15] 8) Evaluate $\iiint_E z\,dV$, where E is the region bounded by the planes $y=0$, $z=0$ $x+y=2$ and the cylinder $y^2+z^2=1$ in the first octant.

Solution. The cylinder $y^2+z^2=1$ is centred on the x axis and intersects the plane $z=0$ in the two lines $y=\pm1$. Viewed from above, the region E is bounded by the lines $y=0$, $x+y=2$ and $y=1$. This base region is pictured on the right below.

z

$x+y=2$

y

x $y^2+z^2=1$

$$\begin{aligned}\iiint_E z\,dV &= \int_0^1 dy \int_0^{2-y} dx \int_0^{\sqrt{1-y^2}} dz\ z \\ &= \int_0^1 dy \int_0^{2-y} dx\ \tfrac{1}{2}z^2\Big|_0^{\sqrt{1-y^2}} \\ &= \int_0^1 dy \int_0^{2-y} dx\ \tfrac{1}{2}(1-y^2) \\ &= \int_0^1 dy\ \tfrac{1}{2}(1-y^2)(2-y) = \tfrac{1}{2}\int_0^1 dy\ \big(2-y-2y^2+y^3\big) \\ &= \tfrac{1}{2}\big[2-\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{1}{4}\big] = \boxed{\tfrac{13}{24}\approx 0.5417}\end{aligned}$$

y

$y=1$

x

$x+y=2$