MATHEMATICS 200 April 2002 Final Exam Solutions

[15] 1) Consider the space curve Γ whose vector equation is

$$\mathbf{r}(t) = t\sin(\pi t)\hat{\imath} + t\cos(\pi t)\hat{\jmath} + t^2\hat{\mathbf{k}} \qquad 0 \le t < \infty$$

This curve starts from the origin and eventually reaches the ellipsoid E whose equation is $2x^2+2y^2+z^2 = 24$.

a) Determine the coordinates of the point P where Γ intersects E.

b) Find the tangent vector of Γ at the point P.

c) Does Γ intersect E at right angles? Why or why not?

Solution. a) The curve intersects E when

$$2\big(t\sin(\pi t)\big)^2 + 2\big(t\cos(\pi t)\big)^2 + \big(t^2\big)^2 = 24 \iff 2t^2 + t^4 = 24 \iff (t^2-4)(t^2+6) = 0$$

Since we need $t > 0$, the desired time is $t = 2$ and the corresponding point is $\mathbf{r}(2) = \boxed{2\hat{\jmath} + 4\hat{\mathbf{k}}}$.

b) Since

$$\mathbf{r}'(t) = \big[\sin(\pi t) + \pi t\cos(\pi t)\big]\hat{\imath} + \big[\cos(\pi t) - \pi t\sin(\pi t)\big]\hat{\jmath} + 2t\hat{\mathbf{k}}$$

a tangent vector to Γ at P is

$$\boxed{\mathbf{r}'(2) = 2\pi\hat{\imath} + \hat{\jmath} + 4\hat{\mathbf{k}}}$$

c) A normal vector to E at P is

$$\boldsymbol{\nabla}(2x^2+2y^2+z^2)\big|_{(0,2,4)} = \big(4x, 4y, 2z\big)\big|_{(0,2,4)} = (0,8,8)$$

Since $\mathbf{r}'(2)$ and $(0,8,8)$ are not parallel, Γ and E $\boxed{\text{do not intersect at right angles}}$.

[10] 2) Let $f(r,\theta) = r^m\cos m\theta$ be a function of r and θ, where m is a positive integer.

a) Find the second order partial derivatives f_{rr}, $f_{r\theta}$, $f_{\theta\theta}$ and evaluate their respective values at $(r,\theta) = (1,0)$.

b) Determine the value of the real number λ so that $f(r,\theta)$ satisfies the differential equation

$$f_{rr} + \tfrac{\lambda}{r}f_r + \tfrac{1}{r^2}f_{\theta\theta} = 0$$

Solution. a) The first order derivatives are

$$f_r(r,\theta) = mr^{m-1}\cos m\theta \qquad f_\theta(r,\theta) = -mr^m\sin m\theta$$

The second order derivatives are

$$f_{rr}(r,\theta) = m(m-1)r^{m-2}\cos m\theta \qquad f_{r\theta}(r,\theta) = -m^2r^{m-1}\sin m\theta \qquad f_{\theta\theta}(r,\theta) = -m^2r^m\cos m\theta$$

so that

$$\boxed{f_{rr}(1,0) = m(m-1),\ f_{r\theta}(1,0) = 0,\ f_{\theta\theta}(1,0) = -m^2}$$

b)

$$f_{rr} + \tfrac{\lambda}{r}f_r + \tfrac{1}{r^2}f_{\theta\theta} = m(m-1)r^{m-2}\cos m\theta + \lambda mr^{m-2}\cos m\theta - m^2r^{m-2}\cos m\theta$$

vanishes for all r and θ if and only if

$$m(m-1) + \lambda m - m^2 = 0 \iff m(\lambda-1) = 0 \iff \boxed{\lambda = 1}$$

[12] 3 a) Show that the function $f(x,y) = 2x + 4y + \frac{1}{xy}$ has exactly one critical point in the first quadrant $x > 0$, $y > 0$, and find its value at that point.

b) Use the second derivative test to classify the critical point in part (a).

c) Hence explain why the inequality $2x + 4y + \frac{1}{xy} \geq 6$ is valid for all positive real numbers x and y.

Solution. a) For $x, y > 0$,

$$\begin{aligned} f_x &= 2 - \tfrac{1}{x^2 y} = 0 \iff y = \tfrac{1}{2x^2} \\ f_y &= 4 - \tfrac{1}{xy^2} = 0 \end{aligned}$$

Subbing the first equation into the second gives $4 - 4x^3 = 0$ which forces $\boxed{x = 1,\ y = \frac{1}{2}}$.

b) The second derivatives are

$$f_{xx}(x,y) = \tfrac{2}{x^3 y} \qquad f_{xy}(x,y) = \tfrac{1}{x^2 y^2} \qquad f_{yy}(x,y) = \tfrac{2}{xy^3}$$

In particular

$$f_{xx}\big(1, \tfrac{1}{2}\big) = 4 \qquad f_{xy}\big(1, \tfrac{1}{2}\big) = 4 \qquad f_{yy}\big(1, \tfrac{1}{2}\big) = 16$$

Since $f_{xx}\big(1, \frac{1}{2}\big) f_{yy}\big(1, \frac{1}{2}\big) - f_{xy}\big(1, \frac{1}{2}\big)^2 = 48 > 0$ and $f_{xx}\big(1, \frac{1}{2}\big) = 4 > 0$, the point $\big(1, \frac{1}{2}\big)$ is a $\boxed{\text{local minimum}}$.

c) As x or y tends to infinity (with the other at least zero), $2x + 4y$ tends to $+\infty$. As x or y tends to zero (with the other bigger than zero), $\frac{1}{xy}$ tends to $+\infty$. Hence as x or y tends to the boundary of the first quadrant (counting infinity as part of the boundary), $f(x,y)$ tends to $+\infty$. As a result $\big(1, \frac{1}{2}\big)$ is a global (and not just local) minimum for f in the first quadrant. Hence $f(x,y) \geq f\big(1, \frac{1}{2}\big) = 6$ for all $x, y > 0$.

[12] 4) Let $f(x,y)$ be a differentiable function with $f(1,2) = 7$. Let

$$\mathbf{u} = \tfrac{3}{5}\hat{\boldsymbol{\imath}} + \tfrac{4}{5}\hat{\boldsymbol{\jmath}}, \qquad \mathbf{v} = \tfrac{3}{5}\hat{\boldsymbol{\imath}} - \tfrac{4}{5}\hat{\boldsymbol{\jmath}}$$

be unit vectors. Suppose it is known that the directional derivatives $D_{\mathbf{u}}f(1,2)$ and $D_{\mathbf{v}}f(1,2)$ are equal to 10 and 2 respectively.

a) Show that the gradient vector $\boldsymbol{\nabla} f$ at $(1,2)$ is $10\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}}$.

b) Determine the rate of change of f at $(1,2)$ in the direction of the vector $\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$.

c) Using the tangent plane approximation, estimate the value of $f(1.01, 2.05)$.

Solution. a) Denote $\boldsymbol{\nabla} f(1,2) = (a,b)$. We are told that

$$\begin{aligned} D_{\mathbf{u}}f(1,2) &= \mathbf{u} \cdot (a,b) = \tfrac{3}{5}a + \tfrac{4}{5}b = 10 \\ D_{\mathbf{v}}f(1,2) &= \mathbf{v} \cdot (a,b) = \tfrac{3}{5}a - \tfrac{4}{5}b = 2 \end{aligned}$$

Adding these two equations gives $\frac{6}{5}a = 12$ which forces $a = 10$ and subtracting the two equations gives $\frac{8}{5}b = 8$, which forces $b = 5$, as desired.

b) The rate of change of f at $(1,2)$ in the direction of the vector $\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$ is

$$\tfrac{\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}}{|\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}|} \cdot \boldsymbol{\nabla} f(1,2) = \tfrac{1}{\sqrt{5}}(1,2) \cdot (10,5) = \boxed{4\sqrt{5} \approx 8.944}$$

c)

$$\begin{aligned} f(1.01, 2.05) &\approx f(1,2) + f_x(1,2) \times (1.01 - 1) + f_y(1,2) \times (2.05 - 2) = 7 + 10 \times 0.01 + 5 \times 0.05 \\ &= \boxed{7.35} \end{aligned}$$

[15] 5) A metal plate is in the form of a semi–circular disc bounded by the x–axis and the upper half of $x^2 + y^2 = 4$. The temperature at the point (x, y) is given by $T(x, y) = \ln\big(1 + x^2 + y^2\big) - y$. Find the coldest point on the plate, explaining your steps carefully. (Note: $\ln 2 \approx 0.693$, $\ln 5 \approx 1.609$)

Solution. The coldest point must be either on the boundary of the plate or in the interior of the plate.

- on the semi–circular part of the boundary $0 \le y \le 2$ and $x^2+y^2 = 4$ so that $T = \ln\big(1+x^2+y^2\big)-y = \ln 5 - y$. The smallest value of $\ln 5 - y$ is taken when $y = 2$ and is $\ln 5 - 2 \approx -0.391$.
- on the flat part of the boundary $y = 0$ and $-2 \le x \le 2$ so that $T = \ln\big(1+x^2+y^2\big)-y = \ln\big(1+x^2\big)$. The smallest value of $\ln\big(1+x^2\big)$ is taken when $x = 0$ and is 0.
- If the coldest point is in the interior of the plate, it must be at a critical point of $T(x, y)$. Since

$$T_x(x, y) = \tfrac{2x}{1+x^2+y^2} \qquad T_y(x, y) = \tfrac{2y}{1+x^2+y^2} - 1$$

a critical point must have $x = 0$ and $\frac{2y}{1+x^2+y^2} - 1 = 0$, which is the case if and only if $x = 0$ and $2y - 1 - y^2 = 0$. So the only critical point is $x = 0,\ y = 1$, where $T = \ln 2 - 1 \approx -0.307$.

Since $-0.391 < -0.307 < 0$, the coldest temperture is -0.391 and the coldest point is $\boxed{(0,2)}$.

[14] 6) Find the dimensions of the box of maximum volume which has its faces parallel to the coordinate planes and which is contained inside the region $0 \le z \le 48 - 4x^2 - 3y^2$.

z

y

x

Solution. The optimal box will have vertices $(\pm x, \pm y, 0)$, $(\pm x, \pm y,\ z)$ with $x, y, z > 0$ and $z = 48 - 4x^2 - 3y^2$. (If the lower vertices are not in the xy–plane, the volume of the box can be increased by lowering the bottom of the box to the xy–plane. If any of the four upper vertices are not on the hemisphere, the volume of the box can be increased by moving the upper vertices outwards to the hemisphere.) The volume of this box will be $(2x)(2y)z$. Use the method of Lagrange multipliers with $f(x, y, z, \lambda) = xyz - \lambda\big(48 - 4x^2 - 3y^2 - z\big)$. Then

$$\begin{aligned} f_x &= yz + 8\lambda x &&= 0 \\ f_y &= xz + 6\lambda y &&= 0 \\ f_z &= xy + \lambda &&= 0 \\ f_\lambda &= 48 - 4x^2 - 3y^2 - z &&= 0 \end{aligned}$$

Multiplying the first equation by x, the second equation by y and the third equation by z gives

$$\begin{aligned} xyz + 8\lambda x^2 &= 0 \\ xyz + 6\lambda y^2 &= 0 \\ xyz + \lambda z &= 0 \end{aligned}$$

This forces $8\lambda x^2 = 6\lambda y^2 = \lambda z$. Since λ cannot be zero (because that would force $xyz = 0$), this in turn gives $8x^2 = 6y^2 = z$. Subbing in to the fourth equation gives

$$48 - \tfrac{z}{2} - \tfrac{z}{2} - z = 0 \implies 2z = 48 \implies z = 24,\ 8x^2 = 24,\ 6y^2 = 24$$

The dimensions of the box of biggest volume are $\boxed{2x = 2\sqrt{3} \text{ by } 2y = 4 \text{ by } z = 24}$.

[12] 7) Consider the volume above the xy–plane that is inside the circular cylinder $x^2+y^2 = 2y$ and underneath the surface $z = 8 + 2xy$.

a) Express this volume as a double integral I, stating clearly the domain over which I is to be taken, Include sketches.

b) Express in Cartesian coordinates, the double integral I as an iterated intergal in two different ways, indicating clearly the limits of integration in each case.

c) How much is this volume?

Solution. a) We are (in part (c)) to find the volume of the set of points (x, y, z) that obey $x^2+(y-1)^2 \le 1$ and $0 \le z \le 8 + 2xy$. When we look at this region from far above we see the set of points (x, y) that obey $x^2 + (y-1)^2 \le 1$ and $8 + 2xy \ge 0$. All points in $x^2 + (y-1)^2 \le 1$ have $-1 \le x \le 1$ and $0 \le y \le 2$ and hence $-2 \le xy \le 2$ and $8 + 2xy \ge 0$. So the domain of integration consists of all of the disk $x^2 + (y-1)^2 \le 1$. This region is sketched on the right below. The volume is

$$\iint_D (8 + 2xy)\; dxdy \text{ where } D = \left\{\ (x, y)\ \middle|\ x^2 + (y-1)^2 \le 1\ \right\}$$

y

D $\quad x^2 + y^2 = 2y$

x

b)

$$\begin{aligned} \text{volume} &= \int_0^2 dy \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} dx\ (8 + 2xy) \\ &= \int_{-1}^1 dx \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} dy\ (8 + 2xy) \end{aligned}$$

c) Since $\iint_D 8\ dxdy$ is just 8 times the area of D, which is π,

$$\begin{aligned} \text{volume} &= 8\pi + \int_0^2 dy \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} dx\ 2xy = 8\pi + 2\int_0^2 dy\ y \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} dx\ x \\ &= \boxed{8\pi} \end{aligned}$$

because $\int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} dx\ x = 0$ for all y, because the integrand is odd and the domain of integration is even.

[10] 8 a) Evaluate $\iiint_\Omega z\, dV$ where Ω is the three dimensional region in the first octant $x \ge 0$, $y \ge 0$, $z \ge 0$, occupying the inside of the sphere $x^2 + y^2 + z^2 = 1$.

b) Use the result in part (a) to quickly determine the centroid of a hemispherical ball given by $z \ge 0$, $x^2 + y^2 + z^2 \le 1$.

Solution. a) In spherical coordinates,

$$x = R\sin\phi\cos\theta \qquad y = R\sin\phi\sin\theta \qquad z = R\cos\phi$$

the sphere $x^2 + y^2 + z^2 = 1$ is $R = 1$, the xy–plane, $z = 0$ is $\phi = \frac{\pi}{2}$, the positive half of the xz–plane, $y = 0$, $x > 0$ is $\theta = 0$ and the positive half of the yz–plane, $x = 0$, $y > 0$ is $\theta = \frac{\pi}{2}$. So

$$\begin{aligned} \iiint_\Omega z\, dV &= \int_0^1 dR \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta\ R^2 \sin\phi (R\cos\phi) \\ &= \tfrac{\pi}{2} \int_0^1 dR \int_0^{\pi/2} d\phi\ R^3 \sin\phi\cos\phi \\ &= \tfrac{\pi}{2} \int_0^1 dR\ R^3\ \tfrac{1}{2}\sin^2\phi\Big|_0^{\pi/2} = \tfrac{\pi}{4}\int_0^1 dR\ R^3 = \boxed{\tfrac{\pi}{16}} \end{aligned}$$

b) The hemispherical ball given by $z \ge 0$, $x^2 + y^2 + z^2 \le 1$ (call it H) has centroid $(\bar{x}, \bar{y}, \bar{z})$ with $\boxed{\bar{x} = \bar{y} = 0}$ (by symmetry) and

$$\bar{z} = \frac{\iiint_H z\, dV}{\iiint_H dV} = \frac{4\iiint_\Omega z\, dV}{\frac{1}{2} \times \frac{4}{3}\pi} = \frac{\frac{\pi}{4}}{\frac{2\pi}{3}} = \boxed{\tfrac{3}{8}}$$