[25] **1.** An antenna at the origin emits a signal whose strength at the point with polar coordinates $[r, \theta]$ is

$$f(r,\theta) = \frac{1+\cos(4\theta)}{r}, \quad r > 0,\ -\frac{\pi}{4} < \theta < \frac{\pi}{4}.$$

(a) Write the level curve $f(r,\theta) = 2$ in polar function form $r = r(\theta)$, $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

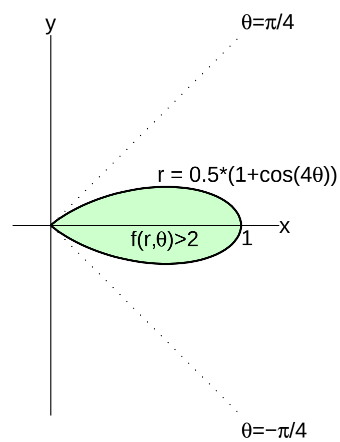
(b) Sketch the region in the xy–plane consisting of all points whose polar coordinates obey the equation $r = r(\theta)$ of part (a). Indicate the region where $f(r,\theta) \geq 2$.

(c) Find the area of the region described in part (b).

(a) The level curve has equation $2 = (1 + \cos(4\theta))/r$. Solving for r gives the polar form:

$$r = r(\theta) = \frac{1+\cos(4\theta)}{2}, \quad -\frac{\pi}{4} < \theta < \frac{\pi}{4}.$$

(b) The curve $r = r(\theta)$ encloses a single lobe along the x-axis. The rightmost point of the lobe is at $(x, y) = (1, 0)$. One has $f(r,\theta) \geq 2$ at points on and inside the closed curve just mentioned.



(c) Call the region $\mathcal{R}$. Its area is

$$\iint_{\mathcal{R}} dA = \int_{-\pi/4}^{\pi/4} \int_0^{\frac{1}{2}(1+\cos(4\theta))} r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4} \frac{r^2}{2}\bigg|_{r=0}^{r=\frac{1}{2}(1+\cos(4\theta))} d\theta$$

$$= \frac{1}{8}\int_{-\pi/4}^{\pi/4} (1+\cos(4\theta))^2 d\theta = \frac{1}{8}\int_{-\pi/4}^{\pi/4} (1 + 2\cos(4\theta) + \cos^2(4\theta))d\theta \stackrel{\text{def}}{=} \frac{1}{8}J.$$

There are several ways to find J. One is to let $u = 4\theta$, $du = 4\,d\theta$:

$$\iint_{\mathcal{R}} dA = \frac{1}{32}\int_{-\pi}^{\pi} (1 + 2\cos u + \cos^2 u)du = \frac{1}{32}\left(u + 2\sin u + \frac{u}{2} + \frac{1}{4}\sin 2u\right)\bigg|_{u=-\pi}^{u=\pi}$$

$$= \frac{1}{32}\left(\frac{3}{2}\pi - \left(-\frac{3}{2}\pi\right)\right) = \frac{3}{32}\pi$$

Or, one could use basic geometry to make three simple observations:

$$\int_{-\pi/4}^{\pi/4} d\theta = \frac{\pi}{2}, \quad \int_{-\pi/4}^{\pi/4} 2\cos(4\theta)\,d\theta = 0, \quad \int_{-\pi/4}^{\pi/4} \cos^2(4\theta)\,d\theta = \frac{\pi}{4}.$$

Summing these values gives $J = 3\pi/4$, so $A = J/8 = 3\pi/32$, as before.

Continued on page 3

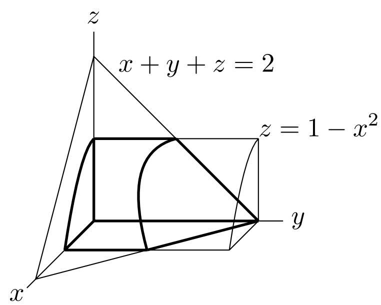
[25] **2.** Let $\mathcal{R}$ denote the solid defined by the system of inequalities

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad z \leq 1 - x^2, \quad x + y + z \leq 2.$$

(a) Express the volume of $\mathcal{R}$ as an iterated triple integral.

(b) Compute the volume of $\mathcal{R}$.

(a) Looking at the figure below from the side (standing far out on the y axis)



we see a base region in the xz–plane consisting of $0 \leq x \leq 1$, $0 \leq z \leq 1 - x^2$. The corresponding triple integral is

$$V = \int_0^1 dx \int_0^{1-x^2} dz \int_0^{2-x-z} dy.$$

(b) The volume is

$$\begin{aligned} V &= \int_0^1 dx \int_0^{1-x^2} dz\ (2 - x - z) \\ &= \int_0^1 dx\ \left[(2-x)(1-x^2) - \tfrac{1}{2}(1-x^2)^2\right] \\ &= \int_0^1 dx\ \left[\frac{3}{2} - x - x^2 + x^3 - \tfrac{1}{2}x^4\right] \\ &= \frac{3}{2} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{10} \\ &= \frac{49}{60} \end{aligned}$$

Continued on page 4

[25] **3.** Let $\mathcal{C}$ be the curve from $P = (1, 0, 0)$ to $Q = (0, \pi/2, \pi/2)$ along the intersection of these surfaces:

$$x = \cos(y), \quad y = z.$$

Choose specific numbers A and B (state your choices clearly!) and then use them to evaluate both

$$I_1 = \int_{\mathcal{C}} (ye^x - Ax^2\cos(z))\,dx + (e^x + By^4z^2)\,dy + (2y^5z - x^3\sin(z))\,dz$$

and $$I_2 = \int_{\mathcal{C}} \left\langle ye^x - Ax^2\cos(z) + 3\sin^2(y),\ e^x + By^4z^2,\ 2y^5z - x^3\sin(z)\right\rangle \bullet d\mathbf{r}.$$

Hint: You can replace A and B with any values you like. Efficient choices would be best; taking $A = 0$ and $B = 0$ is *not efficient at all.*

Both I_1 and I_2 are line integrals of vector fields: $I_1 = \int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$ and $I_2 = I_1 + \int_{\mathcal{C}} \mathbf{G} \bullet d\mathbf{r}$, where

$$\mathbf{F}(x,y,z) = \left\langle ye^x - Ax^2\cos(z),\ e^x + By^4z^2,\ 2y^5z - x^3\sin(z)\right\rangle, \qquad \mathbf{G}(x,y,z) = \left\langle 3\sin^2(y), 0, 0\right\rangle.$$

Line integrals are easy to evaluate when they represent work done by a *conservative* vector field. Could $\mathbf{F}$ be conservative? Only when it passes the screening test, i.e., when

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \text{i.e.,} \quad Ax^2\sin(z) = -3x^2\sin(z), \quad \text{i.e.,} \quad A = -3,$$

and $$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \text{i.e.,} \quad 2By^4z = 10y^4z \quad \text{i.e.,} \quad B = 5.$$

With these choices, $\nabla \times \mathbf{F} \equiv \mathbf{0}$, and it is not hard to see that $\mathbf{F} \equiv \nabla\phi$ for the function

$$\phi(x,y,z) = ye^x + y^5z^2 + x^3\cos(z).$$

Consequently

$$I_1 = \int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \int_{\mathcal{C}} \nabla\phi \bullet d\mathbf{r} = \phi(Q) - \phi(P) = \left[\frac{\pi}{2} + \left(\frac{\pi}{2}\right)^7 + 0\right] - [0 + 0 + 1] = \left(\frac{\pi}{2}\right)^7 + \left(\frac{\pi}{2}\right) - 1.$$

With the same choices for A and B,

$$I_2 = I_1 + \int_{\mathcal{C}} 3\sin^2(y)\,dx.$$

A simple parametrization for $\mathcal{C}$ is given by

$$x = \cos(t),\ y = t,\ z = t,\ 0 \le t \le \pi/2; \quad \text{note} \quad dx = -\sin(t)\,dt,\ dy = dt,\ dz = dt.$$

Hence

$$I_2 = I_1 + 3\int_{t=0}^{\pi/2} \sin^2(t)(-\sin(t)\,dt) = I_1 - \Big[\cos^3(t) - 3\cos(t)\Big]_{t=0}^{\pi/2} = I_1 - 2 = \left(\frac{\pi}{2}\right)^7 + \left(\frac{\pi}{2}\right) - 3.$$

[The integral of $\sin^3(t)$ is given on the formula sheet. One may also write $\sin^3(t) = [1 - \cos^2(t)]\sin(t)$ and then substitute $u = \cos(t)$.]

Continued on page 5

[25] **4.** Let $\mathcal{S}$ be the piece of the paraboloid $z = 10 - x^2 - y^2$ where $1 \le z \le 6$. Compute

$$\iint_{\mathcal{S}} \sqrt{4x^2 + 4y^2 + 1}\, dS.$$

Method 1: Rectangular Coordinates (then switch to polar).

We parametrize S by $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$, where $f(x, y) = 10 - x^2 - y^2$. Then we know that

$$dS = \left(\sqrt{(f_x)^2 + (f_y)^2 + 1}\right) dx\, dy = \left(\sqrt{4x^2 + 4y^2 + 1}\right) dx\, dy.$$

Also note that if $z = 6$ then $r^2 = 4$ and if $z = 1$ then $r^2 = 9$, so we are integrating over an annulus with inner radius 2 and outer radius 3, which we will denote by R. Hence

$$\begin{aligned}
\iint_S \left(\sqrt{4x^2 + 4y^2 + 1}\right) dS &= \iint_R (4x^2 + 4y^2 + 1)\, dx\, dy \\
&= \int_0^{2\pi} \int_2^3 (4r^2 + 1) r\, dr\, d\theta \\
&= 2\pi \int_2^3 (4r^3 + r)\, dr \\
&= 2\pi \Big[r^4 + r^2/2 \Big]_2^3 \\
&= 2\pi[(3^4 + 3^2/2) - (2^4 + 2^2/2)] \\
&= 2\pi(81 + 9/2 - 16 - 2) = \pi(162 + 9 - 32 - 4) = 135\pi.
\end{aligned}$$

Method 2: Cylindrical Coordinates.

We parametrize S in terms of (r, θ) by $\mathbf{s}(r, \theta) = \langle r\cos\theta, r\sin\theta, g(r, \theta)\rangle$, where $g(r, \theta) = 10 - r^2$. Then we know that

$$dS = ((g_\theta)^2 + (rg_r)^2 + r^2)^{1/2}\, dr\, d\theta = (4r^4 + r^2)^{1/2}\, dr\, d\theta.$$

Also note that if $z = 6$ then $r^2 = 4$ and if $z = 1$ then $r^2 = 9$, so we are integrating over an annulus with inner radius 2 and outer radius 3, which we will denote by R. Hence

$$\begin{aligned}
\iint_S \left(\sqrt{4x^2 + 4y^2 + 1}\right) dS &= \iint_R (4r^2 + 1)^{1/2} (4r^4 + r^2)^{1/2}\, dr\, d\theta \\
&= \int_0^{2\pi} \int_2^3 r(4r^2 + 1)\, dr\, d\theta \\
&= 2\pi \int_2^3 (4r^3 + r)\, dr \\
&= 2\pi \Big[r^4 + r^2/2 \Big]_2^3 \\
&= 2\pi[(3^4 + 3^2/2) - (2^4 + 2^2/2)] = 135\pi.
\end{aligned}$$

The End

This examination has 5 pages including this cover

The University of British Columbia

Midterm Examination – 19 Nov 2004

Mathematics 263

Multivariable and Vector Calculus

Closed book examination | Time: 50 minutes

Name______________________________ **Signature** ______________________________

Student Number______________________

Special Instructions:

To receive full credit, all answers must be supported with clear and correct derivations. No calculators, notes, or other aids are allowed. A formula sheet is provided with the test.

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3. Smoking is not permitted during examinations.

1		25
2		25
3		25
4		25
Total		100