[25] **1.** Find and classify all critical points of

$$f(x,y) = x^3 - 3xy^2 + 3x^2 + 3y^2.$$

Calculation gives

$$f(x,y) = x^3 - 3xy^2 + 3x^2 + 3y^2 \qquad \begin{aligned} f_x(x,y) &= 3x^2 - 3y^2 + 6x \\ f_y(x,y) &= -6xy + 6y \end{aligned} \qquad \begin{aligned} f_{xx}(x,y) &= 6x + 6 \\ f_{yy}(x,y) &= -6x + 6 \\ f_{xy}(x,y) &= -6y \end{aligned}$$

At a critical point both $f_x(x,y) = 0$ and $f_y(x,y) = 0$, i.e.,

$$(1) \quad 3(x^2 - y^2 + 2x) = 0, \qquad (2) \quad -6y(x-1) = 0.$$

From equation (2), we get two cases: $x = 1$ or $y = 0$.

Case $x = 1$**.** Here (1) reduces to $y^2 = 3$, so $y = \pm\sqrt{3}$. This gives two CP's:

$$(1, -\sqrt{3}), \qquad (1, \sqrt{3}).$$

Case $y = 0$**.** Here (1) reduces to $0 = x^2 + 2x = x(x+2)$, so $x = 0$ or $x = -2$. This gives two CP's:

$$(0,0), \qquad (-2,0).$$

Here is a table giving the classification of each of the four critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(6) \times (6) - 0^2 > 0$	6	local min
$(-2,0)$	$(-6) \times (18) - 0^2 < 0$		saddle point
$(1,-\sqrt{3})$	$(12) \times 0 - (6\sqrt{3})^2 < 0$		saddle point
$(-1,-\sqrt{3})$	$(12) \times 0 - (-6\sqrt{3})^2 < 0$		saddle point

Continued on page 3

[25] **2.** Consider the equation

$$(*) \qquad 2(x-1)^2 - 2y^2(3-y^2) + (z-1)^2 - 2z^3 + 1 = 0.$$

(a) Assuming that $(*)$ defines z as a function of x and y, find the gradient $\nabla z = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle$.

(b) If $x = 1 + \sqrt{10}\sin t$ and $y = \sqrt{10}\cos t$, use the result in (a) to calculate $\frac{d}{dt} z(x(t), y(t))$ at the point where $(x, y, z) = (1+\sqrt{7}, \sqrt{3}, 2)$.

(a) By implicit differentiation with respect to x, we get:

$$4(x-1) + 2(z-1)\frac{\partial z}{\partial x} - 6z^2 \frac{\partial z}{\partial x} = 0$$

and solving for $\partial z/\partial x$ gives

$$\frac{\partial z}{\partial x} = \frac{2x-2}{3z^2 - z + 1}$$

Similarly, implicit differentiation with respect to y gives:

$$-4y(3-y^2) + 4y^3 + 2(z-1)\frac{\partial z}{\partial y} - 6z^2 \frac{\partial z}{\partial y} = 0$$

and solving for $\partial z/\partial y$ gives

$$\frac{\partial z}{\partial y} = \frac{4y^3 - 6y}{3z^2 - z + 1}$$

This gives the gradient

$$\nabla z(x, y) = \left\langle \frac{2x-2}{3z^2 - z + 1}, \frac{4y^3 - 6y}{3z^2 - z + 1} \right\rangle$$

(b) At the point of interest, we have

$$\frac{dx}{dt} = \sqrt{10}\cos(t), \qquad \frac{dy}{dt} = -\sqrt{10}\sin(t), \qquad \nabla z = \left\langle \frac{2\sqrt{7}}{11}, \frac{6\sqrt{3}}{11} \right\rangle$$

The chain rule says $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$, so at the point of interest,

$$\frac{dz}{dt} = \frac{2\sqrt{7}}{11}\left[\sqrt{10}\cos t\right] + \frac{6\sqrt{3}}{11}\left[-\sqrt{10}\sin t\right]$$

Since the point has $x = 1 + \sqrt{7}$ and $y = \sqrt{3}$, we clearly have $\sqrt{10}\sin t = \sqrt{7}$ and $\sqrt{10}\cos t = \sqrt{3}$. Substituting these values into the above equation, we get

$$\frac{dz}{dt} = \frac{2\sqrt{7}}{11}\sqrt{3} - \frac{6\sqrt{3}}{11}\sqrt{7} = -\frac{4}{11}\sqrt{21}.$$

Continued on page 4

[25] **3.** Consider the set D in the xy-plane defined by

$$D: \qquad x \geq 0,\ y \geq 0,\ x + y \leq 2.$$

Find the maximum value of f on D, and the point(s) where it occurs, given

$$f(x, y) = x^2 y^3 e^{-x-y}.$$

Notice that $f(x,y) = \left(x^2 e^{-x}\right)\left(y^3 e^{-y}\right)$. Use the reduction-of-dimension strategy.

(2D) Interior Points: In the set where $x > 0$, $y > 0$, $x + y < 1$, calculation gives

$$(1) \qquad \frac{\partial f}{\partial x} = y^3 e^{-y}\left[2xe^{-x} - x^2 e^{-x}\right] = x(2-x)y^3 e^{-x-y},$$

$$(2) \qquad \frac{\partial f}{\partial y} = x^2 e^{-x}\left[3y^2 e^{-y} - y^3 e^{-y}\right] = x^2 y^2 (3-y) e^{-x-y}.$$

To get $\partial f/\partial x = 0$ requires either $x = 0$ or $x = 2$ or $y = 0$, but no points in the interior of D satisfy any of these three conditions. So this case produces no points of interest.

(1D) Left Edge: At all points where $x = 0$ and $0 < y < 2$, we have $f(0, y) = 0$.

(1D) Bottom Edge: At all points where $y = 0$ and $0 < x < 2$, we have $f(x, 0) = 0$.

(1D) Top Edge: Here $0 < x < 2$ and $y = 2 - x$, and $f(x, 2-x) = x^2(2-x)^3 e^{-2} \overset{\text{def}}{=} g(x)$. Calculation (product rule) gives
$e^2 g'(x) = [2x](2-x)^3 + x^2[3(2-x)^2(-1)] = x(2-x)^2\left[2(2-x) - 3x\right] = x(2-x)^2(4-5x)$.
The only CP for g obeying $0 < x < 2$ is $x = 4/5$, which corresponds to $(4/5, 6/5)$ on the top edge of D. At this point,

$$f(4/5, 6/5) = \frac{(4^2)(6^3)}{5^5} e^{-2}.$$

(0D) Corner Points: Set D is a triangle, with corners at $(0,0)$, $(2,0)$, $(0,2)$. At each corner point f has the value 0.

Summary: Among all points of interest identified above, the one with the largest function value lies on the top edge of D:

$$\text{Maximum value: } \frac{(4^2)(6^3)}{5^5} e^{-2} = f(4/5, 6/5).$$

Critical Points: To find **all** CP's for f in $\mathbb{R}^2$, use equation (1) to eliminate one variable and study the reduced form of (2). Three cases arise from (1): $x = 0$, $y = 0$, or $x = 2$.

1. If $x = 0$, then (2) holds for all real y. So all points of the form $(0, y)$, $y \in \mathbb{R}$, are CP's.
2. If $y = 0$, then (2) holds for all real x. So all points of the form $(x, 0)$, $x \in \mathbb{R}$, are CP's.
3. If $x = 2$, then (2) holds when either $y = 0$ or $y = 3$. The point $(2, 0)$ has already been catalogued in case 2, but the CP at $(2, 3)$ is new.

Thus f has infinitely many CP's: the two lines $x = 0$ and $y = 0$ and the isolated point $(2, 3)$.

Discussion [not required for credit]: At the maximizing point $(4/5, 6/5)$, there is some constant M such that

$$\nabla f(4/5, 6/5) = \cdots = M \left\langle 1,\, 1 \right\rangle.$$

This is not zero (boundary extrema need not be CP's), but it does point in the outward normal direction to the boundary of D at the point of interest.

Continued on page 5

[25] **4.** Let T be the triangle in the xy-plane bounded by the lines

$$x = 1, \qquad y = 0, \qquad y = x.$$

(a) Let $I = \iint_T f(x,y)\,dA$. Express I as an iterated integral in two different ways: one where the inner integral involves dx, and one where the inner integral involves dy.
(Express your answers in terms of the general function f.)

(b) Given $f(x,y) = e^y/y$, find the average value of f on T.

Recall: The average value of a function f on a plane region T is, by definition,

$$\overline{f} = \frac{1}{\text{Area}(T)} \iint_T f(x,y)\,dA.$$

(a) Projecting T along the x-direction onto the y-axis fills the interval $0 \le y \le 1$; the horizontal filament at level y runs from $x = y$ to $x = 1$. Thus

$$\iint_T f(x,y)\,dA = \int_0^1 \int_y^1 f(x,y)\,dx\,dy$$

Projecting T along the y-direction onto the x-axis fills the interval $0 \le x \le 1$; the vertical filament at position x runs from $y = 0$ to $y = x$. Thus

$$\iint_T f(x,y)\,dA = \int_0^1 \int_0^x f(x,y)\,dy\,dx$$

(b) The region T is a right triangle with both base and height of length 1, so $\text{Area}(T) = 1/2$.

When $f(x,y) = e^y/y$, it is convenient to have an inner integral in terms of x. Thus

$$\iint_T f(x,y)\,dA = \int_0^1 \int_y^1 \frac{e^y}{y}\,dx\,dy = \int_0^1 \left(\frac{1-y}{y}\right) e^y\,dy.$$

This integral cannot be evaluated as a simple formula. The best answer we can give is

$$\overline{f} = \frac{1}{\text{Area}(T)} \iint_T f(x,y)\,dA = 2\int_0^1 \left(\frac{1-y}{y}\right) e^y\,dy.$$

The End

This examination has 5 pages including this cover

The University of British Columbia

Midterm Examination – 28 Oct 2004

Mathematics 263

Multivariable and Vector Calculus

Closed book examination　　　　　　　　　　　　Time: 50 minutes

Name__________________________ **Signature** __________________________

Student Number__________________

Special Instructions:

To receive full credit, all answers must be supported with clear and correct derivations. No calculators, notes, or other aids are allowed. A formula sheet is provided with the test.

Rules governing examinations

1. All candidates should be prepared to produce their library/AMS cards upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

1		25
2		25
3		25
4		25
Total		100