Math 542, Fall 2007, Homework Set 2 (due on Friday October 26 2007)

Instructions

- Homework will be collected at the end of lecture on Friday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, provide self-contained proofs.

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1. Show that $\Gamma(\phi_j u) \to \Gamma_x(u)$ if $\{\phi_j\}$ is a collection of smooth functions with compact support satisfying $\phi_j(x) \neq 0$ and $\mathrm{supp}(\phi_j) \to \{x\}$.
2. Let W be an m-dimensional plane in $\mathbb{R}^n$, $m < n$ $\psi \in C_0^\infty(W)$, and $u = \psi d\sigma$, where $d\sigma$ is the induced Lebesgue measure on W. Find $WF(u)$.
3. For $0 \leq \epsilon \ll 1$, let $u_\epsilon \in \mathcal{D}'(\mathbb{R}^2)$ be defined as follows,
$$\langle u_\epsilon, \varphi\rangle = \int \chi_\epsilon(x_1)\varphi(x_1, \epsilon x_1)\, dx_1, \qquad \chi_\epsilon(\cdot) \in C_0^\infty(\mathbb{R}),$$
so that $\mathrm{supp}(u_\epsilon) \subset \{(x_1, x_2) : x_2 = \epsilon x_1\}$. Is the product $u_0 u_\epsilon$ well-defined as a distribution for all values of ϵ? If so, describe it.
4. Let $P = (x_2 D_{x_2})^2 - D_{x_1}^2 + 2i\mu x_2 D_{x_2}$, $\mu \geq 0$. As always, D_u here stands for the differential operator $i^{-1}\frac{\partial}{\partial u}$.
(a) Show that for every β, the distributions
$$u_\beta(x) = H(x_2)x_2^{\alpha(\beta)}e^{\beta x_1} \in \mathcal{D}'(\mathbb{R}^2)$$
with $\alpha(\beta) = \mu + (\mu^2 + \beta^2)^{\frac{1}{2}}$ satisfy $Pu_\beta = 0$. Here H denotes the Heaviside function.
(b) Show that $WF(u_\beta) = \{(x_1, x_2; \xi_1, \xi_2) : x_2 = \xi_1 = 0,\ \xi_2 \neq 0\}$.
5. Denote by 1_A the characteristic function of a subset A of $\mathbb{R}^n$: $1_A(x) = 1$ for $x \in A$, $1_A(x) = 0$ for $x \notin A$. Determine $WF(u)$ if
(a) $u = 1_A \in \mathcal{D}'(\mathbb{R}^n)$ with $A = \{\varphi(x) > 0\}$, $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $\varphi'(x) \neq 0$ if $\varphi(x) = 0$.
(b) $u = 1_{A_1} - a1_{A_2} \in \mathcal{D}'(\mathbb{R}^2)$, $a \in \mathbb{R}$,
$$A_1 = \{(x_1, x_2) : x_1 > 0, x_2 > 0\},$$
$$A_2 = \{(x_1, x_2) : x_1 < 0, x_2 < 0\}.$$
Discuss the results according to different values of a.

6. Let Ω be a bounded, convex open subset of $\mathbb{R}^2$, with a C^∞ boundary $\partial\Omega = \{\gamma(t) : t \in \mathbb{R}\}$, where $\gamma : \mathbb{R} \to \mathbb{R}^2$ is C^∞, 1-periodic with $\gamma'(t) \neq 0$, $\gamma'(t) \times \gamma''(t) \neq 0$, $\gamma : [0,1) \to \mathbb{R}^2$ injective. Let μ be a measure on $\mathbb{R}^2$ supported by $\partial\Omega$ given by $\mu = \gamma_*(f(t)dt)$, where $f \in C^\infty(\mathbb{R})$ is 1-periodic.
   (a) Study the asymptotics of $\widehat{\mu}(\xi)$ as $|\xi| \to \infty$.
   (b) Study the asymptotics of $\widehat{\chi}(\xi)$, where χ is the characteristic function of Ω.
   (c) Find $WF(u)$.
   (d) Find $WF(\chi)$.
7. Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^3 \geq x_2^2\}$. Let $u = 1_A \in \mathcal{D}'(\mathbb{R}^2)$ be the charactertistic function of A. Choose $\chi = \chi(x_1) \in C^\infty(\mathbb{R}, \mathbb{R})$ with $\chi(x_1) = 1$ if $x_1 < \frac{1}{2}$, $\chi(x_1) = 0$ if $x_1 > 1$. Let $\chi_\epsilon(x_1) = \chi(x_1/\epsilon)$, $\epsilon > 0$.
   (a) For $a \in \mathbb{R}$ and $\lambda > 0$ show that
   $$\widehat{\chi_\epsilon u}(a\lambda, \lambda) = \frac{1}{i\lambda} \int_{-\infty}^{\infty} e^{i\lambda(t^3 - at^2)} \chi_\epsilon(t^2)\, 2t dt.$$
   (b) For $a \neq 0$ and ϵ small enough, find the asymptotics of $\widehat{\chi_\epsilon u}(a\lambda, \lambda)$ as $\lambda \to +\infty$. Show that $(0, 0; a, 1) \in WF(u)$ for every $a \in \mathbb{R}$.
   (c) Determine $WF(u)$.