Decay Estimates for Weighted Oscillatory Integrals in $\mathbb{R}^2$

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ABSTRACT. In this paper, we study decay estimates for a two-dimensional scalar oscillatory integral with degenerate real-analytic phase and amplitude. Integrals such as these form a model for certain higher-dimensional degenerate oscillatory integrals, for which it is known that many of the two-dimensional results fail. We define an analogue of the Newton distance in the weighted case, and prove that this gives the optimal rate of decay for the weighted oscillatory integral under certain generic hypotheses. When these hypotheses fail, we provide counterexamples to show that the optimal rate of decay may be faster in general. We have obtained bounds for the rate of decay in some of these exceptional cases.

1. INTRODUCTION

In this paper, we consider weighted scalar oscillatory integrals in two variables of the form

$$I(\lambda, \varphi) := \int_{\mathbb{R}^2} e^{i\lambda f(x,y)} |g(x,y)|^{\varepsilon} \varphi(x,y)\, dy\, dx, \tag{1.1}$$

where

- f and g are real-analytic, real-valued functions in a neighborhood of the origin in $\mathbb{R}^2$, $f(0,0) = g(0,0) = 0$, $\nabla f(0,0) = (0,0)$,
- ε is a fixed positive number,
- λ is a real parameter, and
- $\varphi \in C_0^{\infty}(\mathbb{R}^2)$.

It is a well-known fact (see [9], [8] or Section 4 of this paper) that if the support of φ is concentrated in a sufficiently small neighborhood of the origin, then the oscillatory integral (1.1) has an asymptotic expansion of the following form as $\lambda \to \infty$:

$$I(\lambda, \varphi) \approx \sum_{p} \sum_{n=0,1} a_{p,n}(\varphi)\lambda^{p}(\ln\lambda)^{n}. \tag{1.2}$$

Here p runs through finitely many arithmetic progressions (independent on φ) whose elements are of the form $-(r + \tilde{r}\varepsilon)$, where r and $\tilde{r}$ are non-negative rationals.

Let us recall the definition of the oscillation index of a scalar oscillatory integral adapted to the weighted situation.

Definition. The oscillation index $\beta(g, f, \varepsilon)$ of the 3-tuple (g, f, ε) at the origin is the maximum of the numbers p having the following property: For any neighborhood V of the origin in $\mathbb{R}^2$, there exists $\varphi \in C_0^\infty(\mathbb{R}^2)$ with support in V such that in the asymptotic expansion for $I(\lambda, \varphi)$ given by (1.2), $a_{p,n}(\varphi) \neq 0$ for some $n = 0, 1$.

The aim of this paper is to compute, to the extent possible, the oscillation index of a two-dimensional weighted oscillatory integral in terms of the Newton diagrams of the amplitude and the phase. The unweighted case (i.e., when $g \equiv 1$) has been considered by Varchenko in [15], where it has been proved that the principal term in the asymptotic expansion is given by the reciprocal of the distance between the origin and the Newton polyhedron of f. However, a counterexample presented in the same paper refutes the hypothesis that the same quantity continues to provide the oscillation index $\beta(1, f, \varepsilon)$ in higher dimensions. The two-dimensional weighted oscillatory integral may be thought of as a model of certain higher-dimensional unweighted oscillatory integrals with special symmetries in the phase (see for instance the introduction of [10]). A case in point is the important counterexample to Arnold's problem given by Varchenko in the context of oscillatory integrals in $\mathbb{R}^3$ (see Section 5, [15]). In our situation, Varchenko's example translates to

$$\iiint_{B^3 \subset \mathbb{R}^3} \exp\{i\lambda[(\mu x_1^2 + x_1^4 + x_2^2 + x_3^2)^2 + x_1^{4p} + (x_2^2 + x_3^2)^{2p}]\} \, dx_1\, dx_2\, dx_3, \tag{1.3}$$

where μ is a real parameter and p is a sufficiently large natural number. Now, a few trivial size estimates coupled with a cylindrical change of co-ordinates transforms the above three-dimensional unweighted integral to a two-dimensional weighted one, given by

$$\iint_{B^2 \subset \mathbb{R}^2} \exp\{i\lambda[(y^2 + x^4 + \mu x^2)^2 + x^{4p} + y^{4p}]\}|y|\, dy\, dx. \tag{1.4}$$

Clearly, the integral in (1.3) has similar decay properties as the integral in (1.4). In general, the hope is that results for integrals of the form (1.1) would shed some light on the behavior of the higher-dimensional unweighted ones they arise from. One therefore expects some of the difficult features of higher dimensions to be reflected in these lower dimensional integrals. It should be mentioned that a special case of the weighted problem arises in the work of Fedoryuk [4], where f is taken to be positive away from the origin and the coordinate axes, and g is a monomial.

In this paper, we make use of Varchenko's technique to obtain an algorithm for computing the oscillation index of the weighted oscillatory integral. In generic situations to be elucidated later, the oscillation index turns out to be the same as the weighted Newton distance. The weighted Newton distance, whose definition is reviewed in Section 1, provides a natural generalization of the notion of distance between the Newton polyhedron and the origin, as originally used by Varchenko. In certain non-generic cases the oscillation index may be strictly larger than the weighted Newton distance, as shown in the example presented in Section 6. This is a significant departure from the unweighted situation in dimension 2. In these cases, the computation of the oscillation index becomes more example-specific, but a refinement of Varchenko's method yields upper and lower bounds of the index even in these cases.

We would like to mention in this context that a problem somewhat related to the one in [15] has been treated by Phong and Stein [12, 13], where the authors obtain the optimal decay rate in L^2 of the unweighted oscillatory integral operator in $\mathbb{R}$ with real-analytic phase. However, the point of emphasis in [12, 13] is somewhat different from [15]. Varchenko's proof for the scalar oscillatory integral is based on successive blow-ups of the phase which reduce the integral to some simple canonical models which can then be treated directly. However, the blow-up processes that he uses are highly non-constructive. On the other hand, the fundamental tool in the method of Phong and Stein is a hands-on decomposition of the complement of the singular variety of a real-analytic function (in their case the Hessian of the phase). A salient feature of our proof is that it combines some of the constructive aspects of Phong and Stein's decomposition technique with the resolution of singularities approach of Varchenko. Such concrete constructions of resolution of singularities have also come up in the recent work of Greenblatt [5, 6], and Rolin, Speissegger and Wilkie [14].

2. Definitions, Notation and Preliminary Observations

In view of the work of Varchenko [15], it is not entirely surprising that the asymptotic behavior of the scalar oscillatory integral (1.1) bears a close connection to the blow-up properties of integrals of the form

$$\int_S \frac{|g(x,y)|^{\varepsilon}}{|f(x,y)|^{\delta}}\, dy\, dx, \tag{2.1}$$

where S is a semi-algebraic set. Finiteness and stability properties of such integrals have been studied in [10, 11] and we shall need many of the ideas from these papers in our analysis.

We begin with a brief review of some of the definitions from [10]. Let $f(x,y) = \sum_{m,n\geq 0} a_{m,n}x^m y^n$ in a neighborhood of the origin.

Definition. Newton's polyhedron of f is defined to be the convex hull of the set

$$\bigcup_{\substack{m,n\geq 0\\ a_{m,n}\neq 0}} \{(x,y) \mid x \geq m,\ y \geq n\}.$$

The union of all compact faces of Newton's polyhedron is called Newton's diagram and is denoted by $\Gamma(f)$.

While the present definition of a Newton diagram is useful from the point of view of computability, there is an alternative description (due to Phong and Stein [12]) which is more useful for analytical purposes and we set up the notation required for this here. By the Weierstrass Preparation Theorem, f and g may be expressed, after a nonsingular change of coordinates, as polynomials in y with coefficients in x, modulo some nonvanishing factors. Factoring out these nonvanishing terms we write f and g as

$$f(x,y) = x^{\tilde{\alpha}_1} y^{\tilde{\beta}_1} \prod_{\nu \mid \nu\in\Xi_f} (y - r_\nu(x)), \tag{2.2a}$$

$$g(x,y) = x^{\tilde{\alpha}_2} y^{\tilde{\beta}_2} \prod_{\mu \mid \mu\in\Xi_g} (y - s_\mu(x)), \tag{2.2b}$$

where $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\beta}_1$, and $\tilde{\beta}_2$ are non-negative integers and $r_\nu(x)$, $s_\mu(x)$ are the non-trivial zeros of f and g respectively. Ξ_f and Ξ_g are index sets that are in one-to-one correspondence with the nontrivial roots of f and g respectively, counted according to multiplicity. In a small neighborhood of the origin these roots admit convergent fractional power series expansions in x, the so-called Puiseux series

$$r_\nu(x) = c_\nu x^{a_\nu} + O(x^{b_\nu}), \quad s_\mu(x) = c_\mu x^{a_\mu} + O(x^{b_\mu}).$$

Here the exponents a_ν, a_μ, b_ν, b_μ are rational numbers and the leading coefficients c_ν, c_μ are nonzero scalars (possibly complex valued). We order the combined set of distinct leading exponents a_ν, a_μ-s into a single increasing sequence a_ℓ, as follows,

$$0 < a_1 < a_2 < \cdots < a_\ell < a_{\ell+1} < \cdots < a_N.$$

The *generalized multiplicity* of f (respectively g) corresponding to a_ℓ, denoted by m_ℓ (respectively n_ℓ), is defined as follows

$$m_\ell := \#\{\nu \mid r_\nu(x) = c_\nu x^{a_\ell} + \cdots, \text{ for some } c_\nu \neq 0\},$$
$$n_\ell := \#\{\mu \mid s_\mu(x) = c_\mu x^{a_\ell} + \cdots, \text{ for some } c_\mu \neq 0\}.$$

If a_ℓ does not occur as a leading exponent of any root of f (respectively g), we set $m_\ell = 0$ (respectively $n_\ell = 0$).

The following quantities arise naturally in the description of the Newton diagrams of f and g:

$$A_\ell = \tilde{\alpha}_1 + a_1 m_1 + a_2 m_2 + \cdots + a_\ell m_\ell, \quad B_\ell = \tilde{\beta}_1 + m_{\ell+1} + \cdots + m_N,$$
$$C_\ell = \tilde{\alpha}_2 + a_1 n_1 + a_2 n_2 + \cdots + a_\ell n_\ell, \quad D_\ell = \tilde{\beta}_2 + n_{\ell+1} + \cdots + n_N,$$

and

$$\delta_\ell = \frac{1 + a_\ell}{A_\ell + a_\ell B_\ell}, \quad \tilde{\delta}_\ell = \frac{1 + a_\ell}{C_\ell + a_\ell D_\ell}. \tag{2.3}$$

In fact, it can be shown that (for a proof see Observation 1 in Section 5(b) of [12]) the Newton diagram of f (respectively g) has vertices at the points (A_ℓ, B_ℓ) (respectively (C_ℓ, D_ℓ)). It follows that the leading exponents a_ℓ of the roots of f (or g) can be read off from its Newton diagram, together with their generalized multiplicities m_ℓ (or n_ℓ). More precisely, the boundary segment in the Newton diagram of f (respectively g) joining the vertices $(A_{\ell-1}, B_{\ell-1})$ and (A_ℓ, B_ℓ) (respectively $(C_{\ell-1}, D_{\ell-1})$ and (C_ℓ, D_ℓ)) has slope $-1/a_\ell$ and vertical height m_ℓ. Furthermore, δ_ℓ^{-1} (respectively $\tilde{\delta}_\ell^{-1}$) is the coordinate of the point of intersection of the above-mentioned boundary line with the bisectrix $y = x$.

It is a point worth noting that if f is subjected to an analytic change of coordinates, then all of the above quantities, namely a_ℓ, m_ℓ, A_ℓ, B_ℓ will possibly change. Here a change of coordinates means a mapping $(x, y) \to (x', y')$ that is analytic with analytic inverse and a nonvanishing Jacobian. However, the concept of a Newton distance continues to make sense in this new set of coordinates, even though it is not invariant under the change. More generally, suppose η is a coordinate change of the form $(x, y) \to (x', y')$, with

$$\begin{cases} x' = x \\ y' = y - q(x) \end{cases} \quad \text{or} \quad \begin{cases} x' = x - q(y) \\ y' = y, \end{cases} \tag{2.4}$$

where q is a convergent real-valued Puiseux series of a single variable in a neighborhood of the origin. Then f and g still have Puiseux factorization representations of the form (2.2). The notions of leading exponent and generalized multiplicity are therefore meaningful and give rise to a convex diagram with vertices (A_ℓ, B_ℓ), via the aforementioned formulas. We will call this diagram the generalized Newton diagram of f in the coordinates η. Coordinate systems η of the form (2.4) are called "good". We denote by C the class of all good coordinate systems. For $\eta \in C$, we write $a_\ell(\eta)$, $m_\ell(\eta)$, $A_\ell(\eta)$, $B_\ell(\eta)$, $\delta_\ell(\eta)$ etc. to accentuate the coordinate-dependence of the relevant quantities. The *weighted Newton distance* of f and g associated to η is defined as follows:

$$\delta_0^w(g, f, \varepsilon; \eta) := \min_\ell \left[\delta_\ell(\eta) \left(1 + \frac{\varepsilon}{\tilde{\delta}_\ell(\eta)} \right) \right],$$

where the index ℓ runs through the combined set of leading exponents of Puiseux series of the roots of f and g expressed in the coordinate system η. The weighted Newton distance plays an important role in the finiteness of the integral in (2.1). In fact, it has been shown in [10] that the integral in (2.1) is finite for $S = B(0; r)$ with sufficiently small $r > 0$ if and only if

$$\delta < \delta_0(g, f, \varepsilon) := \inf_{\eta \in C} \delta_0^w(g, f, \varepsilon; \eta). \tag{2.5}$$

Furthermore, the infimum over C can be replaced by the minimum over a finite subclass $C_0(g, f)$, whose elements can be specified explicitly in terms of the zero variety of f and g. More specifically, $C_0(g, f)$ contains coordinate changes of the form $(x, y) \mapsto (x', y')$, where either $\{y' = y - r(x),\ x' = x\}$ or $\{x' = x - r(y),\ y' = y\}$. Here r is either a real root of fg, or the real part of a complex root of fg. A coordinate system η in $C_0(g, f)$ is called admissible. For details on the description of $C_0(g, f)$ and situations where it arises, see [10]. It is also worth noting that the weighted Newton distance $\delta_0^w(g, f, \varepsilon; \eta)$ or $\delta_0(g, f, \varepsilon)$ are not "real" distances, in the sense that there is no geometrical significance attached to these notions, unlike the standard Newton distance.

Throughout the paper, we use the notation $A \approx B$ to mean that B is the asymptotic expansion of A, and $A \sim B$ to mean that there exists a constant C (possibly depending on f, g and ε) such that $C^{-1}B \le A \le CB$. We are now ready to state and prove our first theorem.

3. STATEMENT OF THEOREM 3.1

Theorem 3.1.

(a) *If $\delta_0(g, f, \varepsilon)$ as defined in (2.5) is not an odd integer, then the oscillation index $\beta(g, f, \varepsilon)$ (defined on page 612) is given by*

$$\beta(g, f, \varepsilon) = -\delta_0(g, f, \varepsilon).$$

(b) *If $\delta_0(g, f, \varepsilon)$ is an odd integer and f does not change sign in a neighborhood of the origin, then the conclusion of part* (a) *holds.*

(c) *Suppose* $\delta_0(g, f, \varepsilon)$ *is an odd integer, which is also a double pole* (*i.e., a pole of order* 2) *of the integral*

$$\int_{B(0;r)} \frac{|g(x,y)|^{\varepsilon}}{|f(x,y)|^{\delta}}\, dy\, dx. \tag{3.1}$$

Here $r > 0$ *is sufficiently small and the integral above is interpreted as a meromorphic function of* δ. *Then the conclusion of part* (a) *holds.*

In terms of the Newton diagram, (*the meromorphic continuation with respect to* δ *of*) *the function given by* (3.1) *has a double pole at* $\delta = \delta_0(g, f, \varepsilon)$ *if and only if there exists a real root of* f, *say* $y - r(x)$ *or* $x - r(y)$, *and an index* ℓ *such that*

$$\delta_0(g, f, \varepsilon) = \frac{1 + \varepsilon D_\ell(\eta)}{B_\ell(\eta)} = \frac{1 + \varepsilon C_\ell(\eta)}{A_\ell(\eta)}, \tag{3.2}$$

where $\eta \in C_0(g, f)$ *is given by*

$$\eta : (x, y) \mapsto (x', y'), \quad \text{where } \begin{cases} x' = x \\ y' = y - r(x) \end{cases} \quad \text{or } \begin{cases} x' = x - r(y) \\ y' = y. \end{cases}$$

An equivalent formulation of (3.2) *is the following:*

$$\delta_0(g, f, \varepsilon) = \delta_\ell(\eta)\left(1 + \frac{\varepsilon}{\tilde{\delta}_\ell(\eta)}\right) = \delta_{\ell+1}(\eta)\left(1 + \frac{\varepsilon}{\tilde{\delta}_{\ell+1}(\eta)}\right). \tag{3.3}$$

Remarks.

(a) Note that the conditions (3.2) and (3.3) have a special geometric significance in the unweighted case, where they are equivalent to the bisectrix intersecting the Newton polygon of f at a vertex. These cases are remarkable in that they provide a necessary condition for the oscillatory integral to have an optimal decay rate of the form $\lambda^{-\beta(g,f,\varepsilon)}(\ln \lambda)$.

(b) In Theorem 7.1, Section 7, we formulate a partial result concerning those cases not covered in Theorem 3.1.

4. The Connection Between I and $I_{\pm}$

Following the analysis of Varchenko, we exploit the connection of the oscillatory integral $I(\lambda, \varphi)$ with the two auxiliary integrals $I_+(\tau, \varphi)$ and $I_-(\tau, \varphi)$ involving the generalized functions $f_{\pm}^{\tau}$:

$$I_+(\tau, \varphi) = \int_{\mathbb{R}^2} (f_+(x,y))^{\tau} |g(x,y)|^{\varepsilon} \varphi(x,y)\, dy\, dx, \tag{4.1}$$

$$I_-(\tau, \varphi) = \int_{\mathbb{R}^2} (f_-(x,y))^{\tau} |g(x,y)|^{\varepsilon} \varphi(x,y)\, dy\, dx. \tag{4.2}$$

Here f_+ and f_- are given by

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0, \end{cases} \qquad f_-(x) = \begin{cases} 0 & \text{iff}(x) \geq 0 \\ -f(x) & \text{iff}(x) < 0. \end{cases}$$

When $\mathrm{Re}(\tau) > 0$, $\tau \in \mathbb{C}$, I_+ and I_- are analytic functions of the parameter τ. It follows from the theorem of Bernstein-Gelfand [3] and of Atiyah [2] that for φ supported in a sufficiently small neighborhood of zero, it is possible to continue I_+ and I_- on $\mathbb{C}$ as meromorphic functions of the parameter τ. Moreover, the poles of I_+ and I_- belong to finitely many arithmetic progressions that do not depend on φ and whose elements are numbers of the form $-(r + \tilde{r}\varepsilon)$, where r and $\tilde{r}$ are non-negative rationals. Given the meromorphic continuation of $I_\pm(\tau, \varphi)$, the proof of the asymptotics of $I(\lambda, \varphi)$ is obtained via the following argument. The connection between these objects is an well-established fact (present in the work of Malgrange [9], Atiyah [2], Gelfand and Shilov [7], Arnold, Gusein-Zadé and Varchenko[1] and Jeanquartier [8]), but we include a proof for the sake of completeness.

Lemma 4.1. *Let the function $I_\pm(\tau, \varphi)$ have poles at the points $-\tau_1$, $-\tau_2$, ..., $-\tau_k$, $(\tau_1 < \tau_2 < \cdots < \tau_k < \cdots)$ and suppose that in a neighborhood of $\tau = -\tau_k$, $I_\pm(\tau, \varphi)$ admits the representation*

$$I_\pm(\tau, \varphi) = \sum_{\ell=1}^{2} \frac{a^\pm_{k,\ell}}{(\tau + \tau_k)^\ell} + \tilde{I}_\pm(\tau, \varphi),$$

where $|a^\pm_{k,1}|^2 + |a^\pm_{k,2}|^2 \neq 0$ and $\tilde{I}_\pm(\tau, \varphi)$ is holomorphic. Then we have the following asymptotic expansion for $I(\lambda, \varphi)$:

$$\begin{aligned} (4.3) \quad I(\lambda, \varphi) \approx \sum_{k=1}^{\infty} \sum_{q=1}^{2} \frac{(-1)^q}{(q-1)!} \Bigg[& a^+_{k,q} \left(\frac{d^{(q-1)}}{ds^{(q-1)}} (\Gamma(s)(-i\lambda)^{-s}) \right)_{s=\tau_k} \\ & + a^-_{k,q} \left(\frac{d^{(q-1)}}{ds^{(q-1)}} (\Gamma(s)(i\lambda)^{-s}) \right)_{s=\tau_k} \Bigg] \end{aligned}$$

for $\lambda \to \infty$.

Remark. It will be clear from the proof that it is not possible for $I_\pm(\tau, \varphi)$ to have a pole of order larger than 2. One sees by integrating by parts that the meromorphic continuation of an integral of the form $F(\tau_1, \tau_2, \dots, \tau_n) = \int_{\mathbb{R}^n} \prod_{i=1}^n x_i^{\tau_i} \varphi(x_1, x_2, \dots, x_n)\, dx_1 \dots dx_n$ can have poles of order no more than n.

Proof of Lemma 4.1. Let us write

$$I(\lambda, \varphi) = J_+(\lambda, \varphi) + J_-(\lambda, \varphi),$$

where

$$J_\pm(\lambda,\varphi) = \int_{\pm f>0} e^{i\lambda f(x,y)}|g(x,y)|^\varepsilon \varphi(x,y)\,dy\,dx. \tag{4.4}$$

By simultaneously "resolving the singularities" of f and g, we first reduce the assertion of Lemma 4.1 to a corresponding statement when

$$f(x,y) = x^{k_1}y^{k_2} \quad \text{and} \quad g(x,y) = x^{m_1}y^{m_2}. \tag{4.5}$$

The statement of the resolution of singularities (adapted to $\mathbb{R}^2$) goes as follows: Let f, $g : \mathbb{R}^2 \to \mathbb{R}$ be analytic functions at the origin, $f(0,0) = g(0,0) = 0$, $\nabla f(0,0) = \nabla g(0,0) = (0,0)$. Then there exist a nonsingular real-analytic two-dimensional manifold Y, and a proper analytic mapping $\pi : Y \to \mathbb{R}^2$ such that at each point of the set $S = \pi^{-1}(0,0)$ there exist local coordinates (x,y) satisfying the following properties:

1. There exist non-negative integers k_1, k_2, m_1, m_2 and smooth functions $\tilde{f}$ and $\tilde{g}$ with $\tilde{f}(0,0), \tilde{g}(0,0) \neq 0$ such that
$$f(\pi(x,y)) = x^{k_1}y^{k_2}\tilde{f}(x,y) \quad \text{and} \quad g(\pi(x,y)) = x^{m_1}y^{m_2}\tilde{g}(x,y).$$
2. The Jacobian of the mapping π has the form
$$J_\pi(x,y) = x^{\ell_1}y^{\ell_2}\tilde{J}_\pi(x,y),$$
where ℓ_1, ℓ_2 are non-negative integers and $\tilde{J}_\pi(0,0) \neq 0$.
3. In a neighborhood of the origin in $\mathbb{R}^2$, π is an analytic isomorphism outside a proper analytic subset in $\mathbb{R}^2$.

Let us choose a smooth finite partition of unity ψ_α such that for any α, there exists an open set containing the support of ψ_α on which conditions 1 and 2 above are satisfied. For $\mathrm{Re}(\tau) > 0$ we have

$$\begin{aligned} I_\pm(\tau,\varphi) &= \int_{\mathbb{R}^2} f_\pm^\tau |g|^\varepsilon \varphi\,dx\,dy = \int_Y (f\circ\pi)_\pm^\tau(\varphi\circ\pi)|g\circ\pi|^\varepsilon\,|J_\pi|\,du\,dv \\ &= \sum_\alpha \int_Y (f\circ\pi)_\pm^\tau(\tilde{\varphi}\circ\pi)|g\circ\pi|^\varepsilon \psi_\alpha |J_\pi|\,du\,dv, \end{aligned}$$

where $\tilde{\varphi} = \varphi\tilde{f}\tilde{g}$, $du\,dv$ is a volume element in Y, and J_π is the Jacobian of the transition from $dx\,dy$ to $du\,dv$. By virtue of conditions 1 and 2 above, the last expression is a sum of terms of the form

$$\int (u^{n_1}v^{n_2})_\pm^\tau |u|^{m_1\varepsilon+\ell_1}\,|v|^{m_2\varepsilon+\ell_2}(\tilde{\varphi}\circ\pi)\psi_\alpha|\tilde{J}_\pi|\,du\,dv.$$

The above integral is again a finite sum of integrals of the type

$$\pm \int_{\substack{\pm u>0 \\ \pm v>0}} |u|^{\tau n_1+m_1\varepsilon+\ell_1} |v|^{\tau n_2+m_2\varepsilon+\ell_2} (\tilde{\varphi} \circ \pi) \psi_\alpha |\bar{J}_\pi| \, du \, dv. \tag{4.6}$$

For any $N \geq 1$ and $\tau > -\max_{i=1,2}(m_i\varepsilon + \ell_i + N)/n_i$, we may formally integrate by parts N times to reduce the integral above to one that is absolutely convergent. Thus, each term in the sum $I_\pm(\tau, \varphi)$ can be analytically continued on $\mathbb{C}$ as a meromorphic function with poles belonging to the arithmetic progressions

$$-\frac{(m_i\varepsilon + \ell_i + k)}{n_i}, \quad k \geq 0,\ k \in \mathbb{Z},\ i = 1, 2.$$

It is therefore sufficient to restrict attention to functions f and g of the special form (4.5) and integrals $I_\pm$ of the form (4.6). Once we have evaluated the meromorphic function in (4.6) at a given $\tau > -\max_{i=1,2}(m_i\varepsilon + \ell_i + N)/n_i$ in terms of a convergent integral using integration by parts, another integration by parts on the integral thus obtained gives a size estimate for this function at the same point. In fact, for any $\varepsilon_0 > 0$, $M \geq 1$ and any τ in the given range such that the distance of the poles of $I_\pm$ from τ is bounded below by ε_0 there exists $C_{M,N,\varepsilon_0} > 0$ such that

$$|I_\pm(\tau, \varphi)| \leq C_{M,N,\varepsilon_0}(1 + |\tau|)^{-M}. \tag{4.7}$$

The proof of Lemma 4.1 is based on the following identities:

$$\int_0^\infty J_+(i\mu, \varphi)\mu^{s-1} \, d\mu = \Gamma(s) I_+(-s, \varphi), \tag{4.8}$$

and

$$J_+(\lambda, \varphi) = \frac{1}{2\pi i} \int_\gamma \Gamma(s)(-i\lambda)^{-s} I_+(-s, \varphi) \, ds, \tag{4.9}$$

where $\operatorname{Im}(\lambda) > 0$ and γ is the contour $c + i\mathbb{R}$, for some small positive number c to be specified. We briefly sketch the proofs of these identities. The equation (4.8) follows by plugging in the expression for J_+ from (4.4) into the left hand side and interchanging the order of integration. In order to see (4.9), we again plug in the defining formula for I_+ from (4.1):

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\gamma \Gamma(s)(-i\lambda)^{-s} I_+(-s,\varphi)\,ds \\
&\quad = \frac{1}{2\pi i}\int_\gamma \Gamma(s)(-i\lambda)^{-s}\Big[\int_{\mathbb{R}^2}(f_+(x,y))^{-s}|g(x,y)|^\varepsilon \varphi(x,y)\,dy\,dx\Big]\,ds \\
&\quad = \int_{\mathbb{R}^2}\Big[\frac{1}{2\pi i}\int_\gamma \Gamma(s)(-i\lambda)^{-s}(f_+(x,y))^{-s}\,ds\Big]|g(x,y)|^\varepsilon \varphi(x,y)\,dy\,dx \\
&\quad = \int_{\mathbb{R}^2}\Big[\sum_{n=0}^\infty \frac{(-1)^n}{n!}(-i\lambda f_+(x,y))^n\Big]|g(x,y)|^\varepsilon \varphi(x,y)\,dy\,dx \\
&\quad = \int_{\mathbb{R}^2} e^{i\lambda f_+(x,y)}|g(x,y)|^\varepsilon \varphi(x,y)\,dy\,dx = J_+(\lambda,\varphi).
\end{aligned}$$

In the penultimate step of the above computation we have used Cauchy's theorem to evaluate the integral, replacing the domain of integration γ by a closed contour of the form

$$\begin{aligned}
&[c-iN, c+iN] \cup [c+iN, -N+iN] \\
&\qquad\qquad \cup [-N+iN, -N-iN] \cup [-N-iN, c-iN],
\end{aligned}$$

and letting $N \to \infty$. The details of the contour integration are left to the interested reader.

Now let us fix a $\lambda > 0$, $\lambda \gg 1$. For $N \geq 1$, we write

$$\begin{aligned}
J_+(\lambda,\varphi) = &\frac{1}{2\pi i}\int_{c+N-i\infty}^{c+N+i\infty} \Gamma(s)(-i\lambda)^{-s} I_+(-s,\varphi)\,ds \\
&+ \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\tilde{\gamma}_{N,R}} \Gamma(s)(-i\lambda)^{-s} I_+(-s,\varphi)\,ds,
\end{aligned}$$

where $\tilde{\gamma}_{N,R}$ is the rectangle in the positive half-plane given by

$$\tilde{\gamma}_{N,R} = \tilde{\gamma}^0_{N,R} + \tilde{\gamma}^1_{N,R} + \tilde{\gamma}^2_{N,R} + \tilde{\gamma}^3_{N,R},$$

oriented in the clockwise direction with

$$\begin{aligned}
\tilde{\gamma}^0_{N,R} &= [c-iR, c+iR], \\
\tilde{\gamma}^1_{N,R} &= [c+iR, N+iR], \\
\tilde{\gamma}^2_{N,R} &= [N+iR, N-iR], \text{ and} \\
\tilde{\gamma}^3_{N,R} &= [N-iR, c-iR].
\end{aligned}$$

We can choose N so that $N \notin \mathbb{N}$ and $\tau_k \neq N$ for any k. For each such $N \geq 1$, there exists k_N such that $\tau_k \in \operatorname{int}(\gamma_N)$ if and only if $1 \leq k \leq k_N$. Observe that by

Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{\tilde{\gamma}_{N,R}} I_+(-s,\varphi)\Gamma(s)(-i\lambda)^{-s}\,ds$$
$$= \sum_{k=1}^{k_N}\sum_{q=1}^{2}\frac{(-1)^{q-1}}{(q-1)!}a^+_{k,q}\left(\frac{d^{q-1}}{ds^{q-1}}(\Gamma(s)(-i)^{-s})\right)_{s=\tau_k},$$

In order to complete the proof we shall show that

(i)

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\tilde{\gamma}^1_{N,R}\cup\tilde{\gamma}^3_{N,R}}\Gamma(s)(-i\lambda)^{-s}I_+(-s,\varphi)\,ds = 0,$$

and

(ii)

$$\left|\int_{c+N-i\infty}^{c+N+i\infty}\Gamma(s)(-i\lambda)^{-s}I_+(-s,\varphi)\,ds\right| \le C_N\lambda^{-N} \quad \text{as } \lambda\to\infty.$$

To see (i), we make use of Stirling's formula and observe that on $\tilde{\gamma}^1_{N,R}$,

$$\begin{aligned}|\Gamma(s)(-i\lambda)^{-s}| &\le C|(\sigma+iR)^{\sigma+iR-1/2}e^{-\sigma-iR}(-i\lambda)^{-\sigma-iR}|\\ &\le C(\sigma^2+R^2)^{(\sigma-1/2)/2}e^{-R\arg(\sigma+iR)}e^{-\sigma}\lambda^{-\sigma}e^{-(\pi/2)R},\end{aligned}$$

where $s=\sigma+iR$, $0<c\le\sigma\le N$. Since

$$0<\arg(\sigma+iR)<\frac{\pi}{2},$$

using (4.7) with $M=0$ gives for $R\ge N$

$$\left|\frac{1}{2\pi i}\int_{\tilde{\gamma}^1_{N,R}}\Gamma(s)(-i\lambda)^{-s}I_+(-s,\varphi)\,ds\right| \le C_N R^N e^{-(\pi/2)R},$$

which approaches zero as R tends to infinity. On $\tilde{\gamma}^3_{N,R}$, we make use of the functional identity of the gamma function

$$\Gamma(x+1)=x\Gamma(x),\qquad x\ne 0,-1,-2,\dots$$

to obtain

$$\begin{aligned}&|\Gamma(s)(-i\lambda)^{-s}|\\ &\quad=|(s-1)(s-2)\cdots(s-N)\Gamma(s-N)(-i\lambda)^{-s}|\\ &\quad\le C(R+N)^N|(\sigma-N-iR)^{\sigma-N-iR-1/2}e^{-\sigma+N+iR}(-i\lambda)^{-\sigma+N+iR}|\\ &\quad\le CR^N((\sigma-N)^2+R^2)^{(\sigma-N-1/2)/2}e^{R\arg(\sigma-N-iR)}e^{-\sigma+N}\lambda^{-\sigma}e^{(\pi/2)R},\end{aligned}$$

where $s = \sigma - iR$, $0 < c \le \sigma \le N$. Now observe that

$$\arg(\sigma - N - iR) < -\frac{\pi}{2},$$

which implies that

$$|e^{-\sigma}\lambda^{-\sigma}e^{R(\pi/2+\arg(\sigma-N-iR))}| \le 1.$$

Furthermore,

$$((\sigma - N)^2 + R^2)^{(\sigma-N-1/2)/2} \le R^{-1/4} \le 1,$$

so that by invoking (4.7) with $M = N + 1$ we get

$$\left| \frac{1}{2\pi i} \int_{\tilde{\gamma}^3_{N,R}} \Gamma(s)(-i\lambda)^{-s} I_+(-s,\varphi)\, ds \right| \le C_N R^{-1},$$

which again approaches zero as R tends to infinity.

The proof of (ii) is very similar. For $s = N + it$, $t > 0$, we have by Stirling's formula

$$|\Gamma(N + it)(-i\lambda)^{-N-it}| \le C\lambda^{-N}(N^2 + t^2)^{(N-1/2)/2} e^{-t\arg(N+it)} e^{-(\pi/2)t}.$$

Using the estimate (4.7) with $M = 0$ we get

$$\left| \int_{c+N+i0}^{c+N+i\infty} \Gamma(s)(-i\lambda)^{-s} I_+(-s,\varphi)\, ds \right| \le C_N \lambda^{-N}.$$

On the other hand, for $s = N - it$, $t > 0$, the functional identity for the gamma function followed by Stirling's formula yields

$$\begin{aligned} |\Gamma(s)(-i\lambda)^{-s}| &\le |(N - it)(N - 1 - it)\dots(c - it)\Gamma(c - it)(-i\lambda)^{-N+it}| \\ &\le C\lambda^{-N}(N + t)^N (c^2 + t^2)^{(c-1/2)/2} e^{t\arg(c-it)} e^{(\pi/2)t}, \end{aligned}$$

where $c = \sup\{N - k < 0 \mid k \in \mathbb{N}\}$. Note that $-1 < 0 < c$. This implies that

$$\arg(c - it) < -\frac{\pi}{2},$$

so that by invoking (4.7) with $M = 2N$, one obtains

$$\left| \int_{c+N-i\infty}^{c+N+i0} \Gamma(s)(-i\lambda)^{-s} I_+(-s,\varphi)\, ds \right| \le C_N \lambda^{-N} \int_0^\infty (1 + |t|)^{-N}\, dt \le C_N \lambda^{-N}.$$

This completes the proof of (ii). In fact, the same proof can be modified to prove the following stronger statement:

$$\left|\left(\frac{d^r}{d\lambda^r}\right)\left[J_+(\lambda,\varphi)-\sum_{k=1}^{k_N}\sum_{q=1}^{2}\frac{(-1)^{q-1}}{(q-1)!}a^+_{k,q}\left(\frac{d^{q-1}}{ds^{q-1}}(\Gamma(s)(-i\lambda)^{-s})\right)_{s=\tau_k}\right]\right|$$
$$\leq C_{N,r}\lambda^{-N-r}, \quad \text{as } \lambda\to\infty.$$

Summarizing, we arrive at the following asymptotic expansion of $J_+(\lambda,\varphi)$,

$$J_+(\lambda,\varphi)\approx\sum_{k=1}^{\infty}\sum_{q=1}^{2}\frac{(-1)^{q-1}}{(q-1)!}a^+_{k,q}\left(\frac{d^{q-1}}{ds^{q-1}}(\Gamma(s)(-i\lambda)^{-s})\right)_{s=\tau_k}.$$

Similarly one obtains the asymptotic expansion of $J_-(\lambda,\varphi)$:

$$J_-(\lambda,\varphi)\approx\sum_{k=1}^{\infty}\sum_{q=1}^{2}\frac{(-1)^{q-1}}{(q-1)!}a^-_{k,q}\left(\frac{d^{q-1}}{ds^{q-1}}(\Gamma(s)(i\lambda)^{-s})\right)_{s=\tau_k}.$$

Combining the two yields the conclusion of Lemma 4.1. ◻

5. PROOF OF THEOREM 3.1

(a). Let us choose $\varphi\geq 0$, $\varphi\equiv 1$ in a neighborhood of the origin. Then by the finiteness result in [10], $\tau=-\delta_0(g,f,\varepsilon)$ is the first pole of

$$\int |g(x,y)|^{\varepsilon}\,|f(x,y)|^{\tau}\varphi(x,y)\,dy\,dx = I_+(\tau,\varphi)+I_-(\tau,\varphi).$$

Since both I_+ and I_- are non-negative functions for $\tau>-\delta_0(g,f,\varepsilon)$, there exists an index q_1 such that

$$a^+_{1,q_1}\geq 0,\quad a^-_{1,q_1}\geq 0,\quad a^+_{1,q_1}+a^-_{1,q_1}>0. \tag{5.1}$$

The principal term in the asymptotic expansion of $I(\lambda,\varphi)$ will be

$$\begin{aligned}&\frac{(\ln\lambda)^{q_1-1}}{\lambda^{\delta_0}}\frac{1}{(q_1-1)!}[\Gamma(\delta_0)e^{(\pi i/2)\delta_0}a^+_{1,q_1}+\Gamma(\delta_0)e^{(-\pi i/2)\delta_0}a^-_{1,q_1}]\\ &\quad=\frac{(\ln\lambda)^{q_1-1}}{\lambda^{\delta_0}}\frac{\Gamma(\delta_0)}{(q_1-1)!}\\ &\qquad\qquad\times\left[(a^+_{1,q_1}+a^-_{1,q_1})\cos\left(\frac{\pi\delta_0}{2}\right)+i(a^+_{1,q_1}-a^-_{1,q_1})\sin\left(\frac{\pi\delta_0}{2}\right)\right],\end{aligned}$$

where we have written δ_0 to mean $\delta_0(g,f,\varepsilon)$. By virtue of (5.1) and the fact that δ_0 is not an odd integer, the above expression does not vanish.

(b). Next suppose that f does not change sign in a neighborhood of the origin, say f is non-negative. Then $I_-(\tau,\varphi) \equiv 0$, which means that $a^-_{1,q_1} = 0$. The principal term in the asymptotic expansion of $I(\lambda,\varphi)$ is therefore

$$\frac{\lambda^{-\delta_0}(\ln\lambda)^{q_1-1}}{(q_1-1)!}\Gamma(\delta_0)e^{(\pi i/2)\delta_0}a^+_{1,q_1} \neq 0.$$

(c). Suppose now that δ_0 is an odd integer and a double pole of the integral in (2.1). Since $q_1 = 2$, there are at most two terms in the asymptotic expansion of $I(\lambda,\varphi)$ of the form $\lambda^{-\delta_0}(\ln\lambda)^k$, namely $k = 0, 1$. These are given by

$$\begin{aligned}
-&\left[a^+_{1,2}\left(\frac{\Gamma'(\delta_0)}{(-i\lambda)^{\delta_0}} - \frac{\Gamma(\delta_0)\ln(-i\lambda)}{(-i\lambda)^{\delta_0}}\right) + a^-_{1,2}\left(\frac{\Gamma'(\delta_0)}{(i\lambda)^{\delta_0}} - \frac{\Gamma(\delta_0)\ln(i\lambda)}{(-i\lambda)^{\delta_0}}\right)\right]\\
&\qquad + \left[a^+_{1,1}\frac{\Gamma(\delta_0)}{(-i\lambda)^{\delta_0}} + a^-_{1,1}\frac{\Gamma(\delta_0)}{(i\lambda)^{\delta_0}}\right]\\
&= \Gamma(\delta_0)\lambda^{-\delta_0}\ln\lambda[a^+_{1,2}e^{(\pi i/2)\delta_0} + a^-_{1,2}e^{-(\pi i/2)\delta_0}]\\
&\qquad + i\frac{\pi}{2}\lambda^{-\delta_0}[-a^+_{1,2}e^{(\pi i/2)\delta_0} + a^-_{1,2}e^{-(\pi i/2)\delta_0}]\\
&\qquad - \Gamma'(\delta_0)\lambda^{-\delta_0}[a^+_{1,2}e^{(\pi i/2)\delta_0} + a^-_{1,2}e^{-(\pi i/2)\delta_0}]\\
&\qquad + \Gamma(\delta_0)\lambda^{-\delta_0}[a^+_{1,1}e^{(\pi i/2)/\delta_0} + a^-_{1,1}e^{-(\pi i/2)\delta_0}]\\
&= \Gamma(\delta_0)\frac{\ln\lambda}{\lambda^{\delta_0}}\left[(a^+_{1,2}+a^-_{1,2})\cos\left(\frac{\pi\delta_0}{2}\right) + i(a^+_{1,2}-a^-_{1,2})\sin\left(\frac{\pi\delta_0}{2}\right)\right]\\
&\qquad + i\frac{\pi}{2}\frac{\Gamma(\delta_0)}{\lambda^{\delta_0}}\left[(a^-_{1,2}-a^+_{1,2})\cos\left(\frac{\pi\delta_0}{2}\right) - i(a^+_{1,2}+a^-_{1,2})\sin\left(\frac{\pi\delta_0}{2}\right)\right]\\
&\qquad - \frac{\Gamma'(\delta_0)}{\lambda^{\delta_0}}\left[(a^+_{1,2}+a^-_{1,2})\cos\left(\frac{\pi\delta_0}{2}\right) + i(a^+_{1,2}-a^-_{1,2})\sin\left(\frac{\pi\delta_0}{2}\right)\right]\\
&\qquad + \frac{\Gamma(\delta_0)}{\lambda^{\delta_0}}\left[(a^+_{1,1}+a^-_{1,1})\cos\left(\frac{\pi\delta_0}{2}\right) + i(a^+_{1,1}-a^-_{1,1})\sin\left(\frac{\pi\delta_0}{2}\right)\right].
\end{aligned}$$

We claim that the oscillation index is δ_0, i.e., the above expression can never be zero. When $a^+_{1,2} \neq a^-_{1,2}$, the growth rate is $\lambda^{-\delta_0}\ln\lambda$. When $a^+_{1,2} = a^-_{1,2}$ (in which case the coefficient of $\lambda^{-\delta_0}\ln\lambda$ vanishes), the above expression has a non-zero real part by virtue of (5.1), namely

$$(-1)^{(\delta_0-1)/2}\frac{\pi}{2}\Gamma(\delta_0)(a^+_{1,2}+a^-_{1,2})\lambda^{-\delta_0}.$$

Thus the oscillation index is δ_0 even in this case. The proof of Theorem 3.1 will therefore be complete if we obtain the characterizations (3.2) and (3.3) of the existence of a double pole at δ_0 in terms of the Newton diagram. For this we set up the following notation for a constructive algorithm for resolution of singularities that will also be useful for a more detailed analysis of I_+ and I_-.

To begin with, let us recall the Puiseux factorization of f given by (2.2) and observe that f can change sign only across a real root. In fact, f changes sign across a real root if and only if the multiplicity of the root is odd. Without loss of generality and for simplicity of exposition let us assume $\tilde{\alpha}_i = \tilde{\beta}_i = 0$, $i = 1, 2$. We fix a small neighborhood V of the origin and order the distinct elements of the set

$$\{\mathrm{Re}(r_\nu) \mid \nu \in \Xi_f\} \cup \{\mathrm{Re}(s_\mu) \mid \mu \in \Xi_g\},$$

on $V \cap \{x > 0\}$, as follows:

$$-1 < q_1(x) < q_2(x) < \cdots < q_R(x) < 1.$$

Here for every i, $1 \le i \le R$, q_i is either a real root of f or g, or the real part of a complex root of f or g. We will denote $q_0 \equiv -1$ and $q_{R+1} \equiv 1$. The roots in $V \cap \{x < 0\}$ can be treated similarly. For $0 \le i \le R$, let

$$\begin{aligned}
\mathcal{R}_i &:= \{(x, y) \in V \mid q_i(x) < y < q_{i+1}(x),\ x > 0\},\\
\mathcal{I}_+ &:= \{i \mid f(x, y) > 0 \text{ for } (x, y) \in \mathcal{R}_i\},\\
\mathcal{I}_- &:= \{i \mid f(x, y) < 0 \text{ for } (x, y) \in \mathcal{R}_i\}.
\end{aligned}$$

For $i \in \mathcal{I}_+$, we define

$$I_+^i(\tau, \varphi) := \int_{\mathcal{R}_i} |f(x, y)|^\tau\, |g(x, y)|^\varepsilon \varphi(x, y)\, dy\, dx.$$

Similarly for $i \in \mathcal{I}_-$, we define $I_+^i(\tau, \varphi)$. Then,

$$I_+(\tau, \varphi) = \sum_{i \in \mathcal{I}_+} I_+^i(\tau, \varphi), \quad I_-(\tau, \varphi) = \sum_{i \in \mathcal{I}_-} I_-^i(\tau, \varphi).$$

Let us consider $I_\pm^i(\tau, \varphi)$ for some $i \in \mathcal{I}_+$. Using the change of variable

$$(x, y) \mapsto (x, u),$$

where

$$u = \frac{y - q_i(x)}{q_{i+1}(x) - q_i(x)},$$

we obtain

$$\begin{aligned}
I_\pm^i(\tau, \varphi) = \int_{\substack{0<x<r\\0<u<1}} &[|f|^\tau\, |g|^\varepsilon \varphi](x, q_i(x) + u(q_{i+1}(x) - q_i(x)))\\
&\times (q_{i+1}(x) - q_i(x))\, du\, dx.
\end{aligned}$$

The change of variable $y \mapsto u$ allows us to express f in the following way:

$$f(x,y) = \prod_{\nu\in\Xi_f} [u(q_{i+1}(x) - q_i(x)) - (r_\nu(x) - q_i(x))].$$

We recall that r_ν-s are the roots of f, and we have made use of the factorization of f given in (2.2). (Without loss of generality we have set $\tilde{\alpha}_i = \tilde{\beta}_i = 0$ for $i = 1$, 2.) Suppose now that $q_{i+1}(x) - q_i(x)$ has leading exponent a. Then the index set Ξ_f may be decomposed into two parts,

$$\Xi^0_f(i,a) := \{\nu \in \Xi_f \mid r_\nu - q_i \text{ has leading exponent strictly smaller than } a\},$$
$$\Xi^1_f(i,a) := \{\nu \in \Xi_f \mid r_\nu - q_i \text{ has leading exponent at least } a\}.$$

Therefore,

$$\begin{aligned}
&|f(x,y)| \\
&= \left| \prod_{\nu\in\Xi^0_f(i,a)} \left[\left(1 - u\frac{q_{i+1}(x)-q_i(x)}{r_\nu(x)-q_i(x)}\right)(r_\nu(x)-q_i(x))\right]\right| \\
&\quad\times \left| \prod_{\nu\in\Xi^1_f(i,a)} \left[\left(u - \frac{r_\nu(x)-q_i(x)}{q_{i+1}(x)-q_i(x)}\right)(q_{i+1}(x)-q_i(x))\right]\right| \\
&= u^{M_i}(1-u)^{M_{i+1}} \left| \prod_{\nu\in\Xi^0_f(i,a)} (r_\nu(x)-q_i(x)) \prod_{\nu\in\Xi^1_f(i,a)} (q_{i+1}(x)-q_i(x))\right| \\
&\quad\times \left| \prod_{\nu\in\Xi^0_f(i,a)} \left(1 - u\frac{q_{i+1}(x)-q_i(x)}{r_\nu(x)-q_i(x)}\right) \prod_{\substack{\nu\in\Xi^1_f(i,a)\\ r_\nu\neq q_i,q_{i+1}}} \left(u - \frac{r_\nu(x)-q_i(x)}{q_{i+1}(x)-q_i(x)}\right)\right|,
\end{aligned}$$

where we have written M_i to denote the multiplicity of the root q_i in f. More precisely, M_i is the positive integer such that

$$f(x,y) = (y - q_i(x))^{M_i}\tilde{f}(x,y),$$

where $\tilde{f}(x, q_i(x)) \not\equiv 0$. Let the multiplicity of q_i in g be denoted by N_i. We observe that by our choice of the q_i-s, $M_i + N_i > 0$ for all i, $1 \le i \le R$. An expression similar to the one above then holds for g with r_ν replaced by s_μ and M_i replaced by N_i. Feeding this back into $I^i_\pm(\tau,\varphi)$ we get:

$$
\begin{aligned}
&I^i_\pm(\tau,\varphi)\\
&= \int_{\substack{0<x<r\\0<u<1}} u^{M_i\tau+N_i\varepsilon}(1-u)^{M_{i+1}\tau+N_{i+1}\varepsilon}(q_{i+1}(x)-q_i(x))\\
&\quad\times\Big|\prod_{\nu\in\Xi^0_f(i,a)}(r_\nu(x)-q_i(x))\prod_{\nu\in\Xi^1_f(i,a)}(q_{i+1}(x)-q_i(x))\Big|^\tau\\
&\quad\times\Big|\prod_{\mu\in\Xi^0_g(i,a)}(s_\mu(x)-q_i(x))\prod_{\mu\in\Xi^1_g(i,a)}(q_{i+1}(x)-q_i(x))\Big|^\varepsilon\\
&\quad\times\Big|\prod_{\nu\in\Xi^0_f(i,a)}\Big(1-u\frac{q_{i+1}(x)-q_i(x)}{r_\nu(x)-q_i(x)}\Big)\prod_{\substack{\nu\in\Xi^1_f(i,a)\\r_\nu\neq q_i,q_{i+1}}}\Big(u-\frac{r_\nu(x)-q_i(x)}{q_{i+1}(x)-q_i(x)}\Big)\Big|^\tau\\
&\quad\times\Big|\prod_{\mu\in\Xi^0_g(i,a)}\Big(1-u\frac{q_{i+1}(x)-q_i(x)}{s_\mu(x)-q_i(x)}\Big)\prod_{\substack{\mu\in\Xi^1_g(i,a)\\s_\mu\neq q_i,q_{i+1}}}\Big(u-\frac{s_\mu(x)-q_i(x)}{q_{i+1}(x)-q_i(x)}\Big)\Big|^\varepsilon\\
&\quad\times\varphi(x,q_i(x)+u(q_{i+1}(x)-q_i(x)))\,du\,dx.
\end{aligned}
$$

Let $\eta_i\in C_0(g,f)$ denote the coordinate transformation given by:

$$(x,y)\mapsto(x,y-q_i(x)).$$

Recalling the definition of $a_\ell(\eta_i)$ introduced in Section 2, note that there exists ℓ such that

$$a_\ell(\eta_i)=a.$$

Using the definitions of $\Xi^0_f(i,a)$ and $\Xi^1_f(i,a)$, we observe that

$$\Big|\prod_{\nu\in\Xi^0_f(i,a)}(r_\nu(x)-q_i(x))\prod_{\nu\in\Xi^1_f(i,a)}(q_{i+1}(x)-q_i(x))\Big|\sim x^{A_\ell(\eta_i)+a_\ell(\eta_i)B_\ell(\eta_i)},$$

from which one obtains

$$
\begin{aligned}
&(q_{i+1}(x)-q_i(x))\times\Big|\prod_{\nu\in\Xi^0_f(i,a)}(r_\nu(x)-q_i(x))\prod_{\nu\in\Xi^1_f(i,a)}(q_{i+1}(x)-q_i(x))\Big|^\tau\\
&\qquad\times\Big|\prod_{\mu\in\Xi^0_g(i,a)}(s_\mu(x)-q_i(x))\prod_{\mu\in\Xi^1_g(i,a)}(q_{i+1}(x)-q_i(x))\Big|^\varepsilon\\
&\quad\sim x^{[A_\ell(\eta_i)+a_\ell(\eta_i)B_\ell(\eta_i)]\tau+[C_\ell(\eta_i)+a_\ell(\eta_i)D_\ell(\eta_i)]\varepsilon+a_\ell(\eta_i)}.
\end{aligned}
$$

The meromorphic nature of the integral $I^i_\pm(\tau,\varphi)$ will be specified by the singularities of the integrand on the domain on integration. In our case, these are at

$u = 0$, $u = 1$ and $x = 0$. In order to obtain the location of the poles and the coefficients of the singular terms, we recall a theorem of Gelfand and Shilov [7], which states that if θ is a smooth cut-off on $\mathbb{R}$ supported in $[-1, 1]$ and non-vanishing at the origin, then the first pole of the integral

$$\int_0^1 x^\tau \theta(x)\, dx$$

occurs at $\tau = -1$, the order of the pole is simple and the residue is $\theta(0)$. A natural generalization of the above theorem to higher dimensions is the following lemma (see for instance [15])

Lemma 5.1. *Let $\psi(y_1, y_2, \ldots, y_k, \mu)$ be a compactly supported infinitely differentiable function in the variables y_1, y_2, …, y_k that is meromorphic in the parameter $\mu \in \mathbb{C}^\ell$. Then the function*

$$\mathcal{U}(\tau_1, \tau_2, \ldots, \tau_k, \mu) = \int_{y_1,\ldots,y_k>0} \Big(\prod_{i=1}^{k} y_i^{\tau_i}\Big) \psi(y_1, y_2, \ldots, y_k, \mu)\, dy_1\, dy_2 \cdots dy_k$$

can be meromorphically continued for all values of τ_1, τ_2, …, τ_k and its poles are either those already possessed by the function ψ or they can lie only on hyperplanes of the form $\tau_i + s = 0$, where $s \in \mathbb{N}$.

In our case the above result yields the following conclusion which we state in the form of a lemma.

Lemma 5.2. *If $\delta_0(g, f, \varepsilon)$ is a double pole of $I^i_\pm(\tau, \varphi)$, then there exists ℓ such that*

$$\delta_0(g, f, \varepsilon) = \delta_\ell(\eta_i)\left(1 + \frac{\varepsilon}{\bar{\tilde{\delta}}_\ell(\eta_i)}\right) = \delta_{\ell+1}(\eta_i)\left(1 + \frac{\varepsilon}{\bar{\tilde{\delta}}_{\ell+1}(\eta_i)}\right).$$

Proof. We first observe that we cannot directly apply the theorem of Gelfand and Shilov to the integral $I^i_\pm(\tau, \varphi)$. This is because if we write $I^i_\pm(\tau, \varphi)$ in the form

$$I^i_\pm(\tau, \varphi) = \int_{\substack{0<x<r\\0<u<1}} u^{M_i\tau+N_i\varepsilon}(1-u)^{M_{i+1}\tau+N_{i+1}\varepsilon} x^{(A_\ell+a_\ell B_\ell)\tau+(C_\ell+a_\ell D_\ell)\varepsilon+a_\ell} \times \theta(x, u)\, du\, dx,$$

where a_ℓ, A_ℓ etc. are computed in the coordinates η_i, then the cutoff function θ involves factors of the form

$$\Big| \prod_{\substack{\nu\in\Xi^1_f(i,a)\\ r_\nu\neq q_i, q_{i+1}}} \left(u - \frac{r_\nu(x) - q_i(x)}{q_{i+1}(x) - q_i(x)}\right)\Big|^\tau \tag{5.2}$$

and

$$x = 0 \Big| \prod_{\substack{\mu \in \Xi_g^1(i,a) \\ s_\mu \neq q_i, q_{i+1}}} \left(u - \frac{s_\mu(x) - q_i(x)}{q_{i+1}(x) - q_i(x)} \right) \Big|^{\varepsilon}, \tag{5.3}$$

which are not always smooth. We explore the nature of these factors in some detail. Since there is no root of f or g whose real part lies in between q_i and q_{i+1}, none of the terms

$$u - \mathrm{Re}\left(\frac{r_\nu(x) - q_i(x)}{q_{i+1}(x) - q_i(x)} \right) \quad \text{or} \quad u - \mathrm{Re}\left(\frac{s_\mu(x) - q_i(x)}{q_{i+1}(x) - q_i(x)} \right) \tag{5.4}$$

vanish on the domain of integration $\{(x,u) \mid 0 < x < r,\ 0 < u < 1\}$ except possibly at $(u,x) = (0,0)$ or $(u,x) = (1,0)$. The following cases may arise.

(1). For some $\nu \in \Xi_f^1(i,a)$ or $\mu \in \Xi_g^1(i,a)$, r_ν, $s_\mu \neq q_i$, q_{i+1}, the root

$$\frac{r_\nu(x) - q_i(x)}{q_{i+1}(x) - q_i(x)} \quad \text{or} \quad \frac{s_\mu(x) - q_i(x)}{q_{i+1}(x) - q_i(x)} \tag{5.5}$$

is real-valued, with leading exponent 0. This occurs if $r_\nu - q_i$ (or $s_\mu - q_i$) has leading exponent a. In this case, the leading coefficient must be either < 0 or ≥ 1.

(a) If the leading coefficient is $\neq 1$, the expressions (5.2) and (5.3) can be made nonvanishing on the domain of integration by choosing r sufficiently small, and hence smooth for all values of ε and/or τ. The corresponding factor can then be absorbed into the cutoff φ.
(b) If the leading coefficient is equal to 1, then the corresponding root in (5.5) is given by the Puiseux expansion $1 + cx^b +$ higher order terms, where $b > 0$ and $c > 0$. Thus for r sufficiently small, the only point in $\{0 < x < r, 0 < u < 1\}$ where the corresponding term given in (5.4) can vanish is $(u,x) = (1,0)$. In this case, the factor in (5.2) blows up at $(u,x) = (1,0)$ for negative τ. The factor in (5.3) is also non-smooth in general at the specified point, unless $(s_\mu - q_i)/(q_{i+1} - q_i)$ is analytic and ε is an even integer.

(2). Suppose next that the root in (5.5) is real-valued with leading exponent > 0. Then the leading coefficient must be < 0, from which we conclude that the only point where (5.4) vanishes is $u = 0$, $x = 0$. By the same reasoning as above, the corresponding factors (5.2) and (5.3) are non-smooth in general in these cases.

(3). The treatment of the case when the roots in (5.2) and (5.3) are complex-valued is essentially a repetition of the real-valued case. We write

$$\left|u - \frac{r_\nu(x) - q_i(x)}{q_{i+1}(x) - q_i(x)}\right| \sim \left|u - \mathrm{Re}\left(\frac{r_\nu(x) - q_i(x)}{q_{i+1}(x) - q_i(x)}\right)\right| + \left|\mathrm{Im}\left(\frac{r_\nu(x) - q_i(x)}{q_{i+1}(x) - q_i(x)}\right)\right|, \tag{5.6}$$

and consider the following subcases.

(a) First let us suppose that the Puiseux series of $\mathrm{Re}[(r_\nu - q_i)/(q_{i+1} - q_i)]$ has the same leading exponent as $(r_\nu - q_i)/(q_{i+1} - q_i)$. If this leading exponent is > 0, then the imaginary part of the expression in (5.6) also had leading exponent > 0 and vanishes if and only if $x = 0$. Consequently the real part vanishes only when $u = 0$. This gives rise to the same problems with smoothness as discussed earlier. If the leading exponent is 0, then the leading coefficient of the Puiseux series of $\mathrm{Re}(r_\nu - q_i)/(q_{i+1} - q_i)$ is either < 0 or ≥ 1. If the leading coefficient is $\neq 1$, the expression in (5.6) is nonvanishing, hence (5.2) and (5.3) are smooth. If the leading coefficient is 1, the second coefficient must be positive as before, which means that (5.6) vanishes if and only if $(u, x) = (1, 0)$.

(b) Next let the leading exponent of $\mathrm{Re}(r_\nu - q_i)/(q_{i+1} - q_i)$ be larger than that of $(r_\nu - q_i)/(q_{i+1} - q_i)$. If the leading exponent of $(r_\nu - q_i)/(q_{i+1} - q_i)$ is 0, the expression (5.6) is already nonvanishing by virtue of its imaginary part, hence smooth. If the leading exponent is > 0, the imaginary part vanishes when $x = 0$, and the real part when $u = 0$.

In the remainder of this section, we rewrite $I^i_\pm(\tau, \varphi)$ as a sum of integrals, each of which can be treated by the theorem of Gelfand and Shilov. To begin with, we decompose the domain of integration in $I^i_\pm(\tau, \varphi)$ as follows:

$$\{(u, x) \mid 0 < u < 1,\ 0 < x < r\} = \left[\left(0, \frac{1}{2}\right) \times (0, r)\right] \cup \left[\left(\frac{1}{2}, 1\right) \times (0, r)\right],$$

and write $I^i_\pm(\tau, \varphi)$ as a sum of two integrals, one over each domain:

$$I^i_\pm(\tau, \varphi) = I^i_{\pm,0}(\tau, \varphi) + I^i_{\pm,1}(\tau, \varphi).$$

For $I^i_{\pm,1}(\tau, \varphi)$, whose domain of integration is the region $(\frac{1}{2}, 1) \times (0, r)$, we make the change of variable $(x, u) \mapsto (x', u')$, where $x' = x$, $u' = 1 - u$, to get an integral whose domain of integration is $(0, \frac{1}{2}) \times (0, r)$ and whose integrand has the same form as that of $I^i_{\pm,0}(\tau, \varphi)$. It is therefore sufficient to work with $I^i_{\pm,0}(\tau, \varphi)$.

Let us denote by

$$b_1 < b_2 < \cdots < b_k < \cdots < b_K, \quad K = K_i \text{ dependent on } i,$$

the distinct leading exponents of the following collection of Puiseux series:

$$\left\{\frac{r_\nu(x)-q_i(x)}{q_{i+1}(x)-q_i(x)} \mid r_\nu \in \Xi^1_f(i,a),\ r_\nu \neq q_i, q_{i+1}\right\} \\ \cup \left\{\frac{s_\mu(x)-q_i(x)}{q_{i+1}(x)-q_i(x)} \mid s_\mu \in \Xi^1_g(i,a),\ s_\mu \neq q_i, q_{i+1}\right\}.$$

Depending on the size of these factors, we decompose the integral $I^i_{\pm,0}(\tau,\varphi)$ into a sum of integrals: $I^i_{\pm,0}(\tau,\varphi) = \sum_{k=0}^K I^{i,k}_{\pm,0}(\tau,\varphi)$, where $I^{i,k}_{\pm,0}(\tau,\varphi)$ is the integral with integrand same as $I^i_{\pm,0}(\tau,\varphi)$ but domain of integration equal to

$$\begin{array}{ll} \{0<x<r,\ 0<u<x^{b_K}\} & \text{if } k=K, \\ \{0<x<r,\ x^{b_{k+1}}<u<x^{b_k}\} & \text{if } 1\le k\le K-1, \\ \{0<x<r,\ x^{b_1}<u<\frac{1}{2}\} & \text{if } k=0. \end{array}$$

We will analyze each of these integrals separately.

Let us first consider $I^{i,k}_{\pm,0}(\tau,\varphi)$ for $1\le k\le K-1$. We make a change of variable $(x,u)\mapsto(x_1,u_1)$, where $x_1=x$, $u_1=u/x^{b_k}$, which reduces the domain of integration to $\{0<x_1<r,\ x_1^{b_{k+1}-b_k}<u_1<1\}$. Rewriting $I^{i,k}_{\pm,0}(\tau,\varphi)$ after the change of variable, we obtain

$$\begin{aligned} I^{i,k}_{\pm,0}(\tau,\varphi) = \int_{\substack{0<x<r \\ x^{b_{k+1}-b_k}<u_1<1}} & (u_1x^{b_k})^{M_i\tau+N_i\varepsilon}(q_{i+1}(x)-q_i(x))x^{b_k} \\ \times & \Bigl|\prod_{\nu\in\Xi^0_f(i,a)}(r_\nu(x)-q_i(x))\prod_{\nu\in\Xi^1_f(i,a)}(q_{i+1}(x)-q_i(x))\Bigr|^\tau \\ \times & \Bigl|\prod_{\mu\in\Xi^0_g(i,a)}(s_\mu(x)-q_i(x))\prod_{\mu\in\Xi^1_g(i,a)}(q_{i+1}(x)-q_i(x))\Bigr|^\varepsilon \\ \times & \Bigl|\prod_{\nu\in\Xi^0_f(i,a)}\Bigl(1-u_1x^{b_k}\frac{q_{i+1}(x)-q_i(x)}{r_\nu(x)-q_i(x)}\Bigr)\prod_{\substack{\nu\in\Xi^1_f(i,a) \\ r_\nu\neq q_i,q_{i+1}}}\Bigl(u_1x^{b_k}-\frac{r_\nu(x)-q_i(x)}{q_{i+1}(x)-q_i(x)}\Bigr)\Bigr|^\tau \\ \times & \Bigl|\prod_{\mu\in\Xi^0_g(i,a)}\Bigl(1-u_1x^{b_k}\frac{q_{i+1}(x)-q_i(x)}{s_\mu(x)-q_i(x)}\Bigr)\prod_{\substack{\mu\in\Xi^1_g(i,a) \\ s_\mu\neq q_i,q_{i+1}}}\Bigl(u_1x^{b_k}-\frac{s_\mu(x)-q_i(x)}{q_{i+1}(x)-q_i(x)}\Bigr)\Bigr|^\varepsilon \\ & \times\varphi(x,q_i(x)+u_1x^{b_k}(q_{i+1}(x)-q_i(x)))\,du_1\,dx \quad = \end{aligned}$$

$$
\begin{aligned}
&= \int_{\substack{0<x<r\\ x^{b_{k+1}-b_k}<u_1<1}} u_1^{M_i\tau+N_i\varepsilon}(q_{i+1}(x)-q_i(x))(x^{b_k})^{M_i\tau+N_i\varepsilon+1}\\
&\quad\times\Bigg|\prod_{\nu\in\Xi_f^0(i,b_k+a)}(r_\nu(x)-q_i(x))\prod_{\substack{\nu\in\Xi_f^1(i,b_k+a)\\ r_\nu\neq q_i}}x^{b_k}(q_{i+1}(x)-q_i(x))\Bigg|^{\tau}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times(q_{i+1}(x)-q_i(x))^{M_i\tau}\\
&\quad\times\Bigg|\prod_{\mu\in\Xi_g^0(i,b_k+a)}(s_\mu(x)-q_i(x))\prod_{\substack{\mu\in\Xi_g^1(i,b_k+a)\\ s_\mu\neq q_i}}x^{b_k}(q_{i+1}(x)-q_i(x))\Bigg|^{\varepsilon}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times(q_{i+1}(x)-q_i(x))^{N_i\varepsilon}\\
&\quad\times\Bigg|\prod_{\nu\in\Xi_f^0(i,b_k+a)}\left(1-u_1x^{b_k}\frac{q_{i+1}(x)-q_i(x)}{r_\nu(x)-q_i(x)}\right)\\
&\qquad\qquad\times\prod_{\substack{\nu\in\Xi_f^1(i,b_k+a)\\ r_\nu\neq q_i}}\left(u_1-x^{-b_k}\frac{r_\nu(x)-q_i(x)}{q_{i+1}(x)-q_i(x)}\right)\Bigg|^{\tau}\\
&\quad\times\Bigg|\prod_{\mu\in\Xi_g^0(i,b_k+a)}\left(1-u_1x^{b_k}\frac{q_{i+1}(x)-q_i(x)}{s_\mu(x)-q_i(x)}\right)\\
&\qquad\qquad\times\prod_{\substack{\mu\in\Xi_g^1(i,b_k+a)\\ s_\mu\neq q_i}}\left(u_1-x^{-b_k}\frac{s_\mu(x)-q_i(x)}{q_{i+1}(x)-q_i(x)}\right)\Bigg|^{\varepsilon}\\
&\qquad\qquad\qquad\qquad\times\varphi(x,q_i(x)+u_1x^{b_k}(q_{i+1}(x)-q_i(x)))\,du_1\,dx.
\end{aligned}
$$

We observe that there exists $\nu\in\Xi_f^1(i,a)$, $r_\nu\neq q_i$, q_{i+1}, such that the leading exponent of $r_\nu-q_i$ is b_k+a. This is because if $(r_\nu-q_i)/(q_{i+1}-q_i)$ has leading exponent b_k, then $r_\nu-q_i$ must have leading exponent b_k+a. Thus there exists an index m such that $a_m(\eta_i)=b_k+a$. Note that

$$
\begin{aligned}
&(q_{i+1}(x)-q_i(x))^{M_i\tau+N_i\varepsilon+1}(x^{b_k})^{M_i\tau+N_i\varepsilon+1}\\
&\qquad\times\Bigg|\prod_{\nu\in\Xi_f^0(i,b_k+a)}(r_\nu(x)-q_i(x))\prod_{\substack{\nu\in\Xi_f^1(i,b_k+a)\\ r_\nu\neq q_i}}x^{b_k}(q_{i+1}(x)-q_i(x))\Bigg|^{\tau}\\
&\qquad\times\Bigg|\prod_{\mu\in\Xi_g^0(i,b_k+a)}(s_\mu(x)-q_i(x))\prod_{\substack{\mu\in\Xi_g^1(i,b_k+a)\\ s_\mu\neq q_i}}x^{b_k}(q_{i+1}(x)-q_i(x))\Bigg|^{\varepsilon}\\
&\quad\sim x^{(Am+amBm)\tau+(Cm+amDm)\varepsilon+am},
\end{aligned}
$$

where all the numbers a_m, A_m, etc. have been computed based on the coordinates η_i. We need one more change of variable to reduce the integral to standard form. Let us consider the map $(x, u_1) \mapsto (x_1, u_2)$, where

$$x_1 = x u_1^{-1/(b_{k+1}-b_k)}, \quad u_2 = u_1. \tag{5.7}$$

The domain of integration changes to

$$\begin{aligned}&\{0 < u_1 < 1,\ 0 < x_1 < \min(r u_1^{-1/(b_{k+1}-b_k)}, 1)\}\\ &\qquad = \{0 < u_1 < r^{b_{k+1}-b_k},\ 0 < x_1 < 1\} \cup \{r^{b_{k+1}-b_k} < u_1 < 1,\ 0 < x_1 < 1\}.\end{aligned}$$

We accordingly write $I^{i,k}_{\pm,0}(\tau,\varphi) = I^{i,k,1}_{\pm,0}(\tau,\varphi) + I^{i,k,2}_{\pm,0}(\tau,\varphi)$, depending on whether the integral is over the first or second domain given above. The integral $I^{i,k,2}_{\pm,0}(\tau,\varphi)$ cannot have a double pole, since u_1 is bounded away from 0 on its domain of integration. We therefore only concentrate on $I^{i,k,1}_{\pm}(\tau,\varphi)$. For $\nu \in \Xi^1_f(i, b_k + a)$, the change of variable (5.7) yields

$$\begin{aligned}&u_1 - x^{-b_k}\frac{r_\nu(x) - q_i(x)}{q_{i+1}(x) - q_i(x)}\\ &\qquad = u_1 - c_\nu x^{b_\ell - b_k} - \text{higher order terms}\\ &\qquad = u_1 - c_\nu (x_1 u_1^{1/(b_{k+1}-b_k)})^{b_\ell - b_k} - \text{higher order terms}\\ &\qquad = u_1[1 - c_\nu x_1^{b_\ell - b_k} u_1^{(b_\ell - b_k)/(b_{k+1}-b_k)-1} - \text{higher order terms}],\end{aligned}$$

for some $b_\ell \ge b_k$ and $c_\nu \ne 0$. First suppose that $b_\ell > b_k$. Then for $0 < u_1 < r^{b_{k+1}-b_k}$ and r sufficiently small, the factor accompanying u_1 is nonvanishing on the domain of integration. This is immediate if $b_\ell > b_{k+1}$. If $b_\ell = b_{k+1}$, we recall that $c_\nu < 0$ and use the next term of the Puiseux series:

$$\begin{aligned}&u_1 - x^{-b_k}\frac{r_\nu(x) - q_i(x)}{q_{i+1}(x) - q_i(x)}\\ &\qquad = u_1 - c_\nu x^{b_{k+1} - b_k} - d_\nu x^{b - b_k} - \text{higher order terms}\\ &\qquad = u_1[1 - c_\nu x_1^{b_{k+1} - b_k} - d_\nu x_1^{b-b_k} u_1^{(b - b_k)/(b_{k+1}-b_k)-1} - \cdots],\end{aligned}$$

for some $b > b_{k+1}$. We note that

$$\begin{aligned}&[1 - c_\nu x_1^{b_{k+1} - b_k} - d_\nu x_1^{b-b_k} u_1^{(b - b_k)/(b_{k+1}-b_k)-1} - \cdots]\\ &\qquad\qquad \ge 1 - 2|d_\nu| r^{(b-b_k)/(b_{k+1}-b_k)-1} > 0.\end{aligned}$$

Thus the expression $u_1 - x^{-b_k}(r_\nu - q_i)/(q_{i+1} - q_i)$ essentially contributes a single u_1 for every ν with the property that $r_\nu - q_i$ has leading exponent $b_\ell + a$, $b_\ell > b_k$. Next let us consider the case $b_\ell = b_k$. Since there is no root of f or g whose real part lies between q_i and q_{i+1}, $\mathrm{Re}(c_\nu)$ is either < 0 or ≥ 1. Since $u_1 \leq r^{b_{k+1}-b_k}$ and r may be chosen sufficiently small, the expression $[u_1 - c_\nu - \cdots]$ is nonvanishing. We therefore have

$$
\begin{aligned}
(5.8)\quad I_{\pm,0}^{i,k,1}(\tau,\varphi) &= \int_{0<x_1<1,\ 0<u_1<r^{b_{k+1}-b_k}} u_1^{B_m\tau + D_m\varepsilon} \\
&\quad \times (x_1 u_1^{1/(b_{k+1}-b_k)})^{(A_m+a_mB_m)\tau + (C_m+a_mD_m)\varepsilon + a_m} \\
&\qquad\qquad \times \varphi(x_1 u_1^{1/(b_{k+1}-b_k)}, q_i(x_1 u_1^{1/(b_{k+1}-b_k)}) \\
&\quad + u_1^{b_{k+1}/(b_{k+1}-b_k)} x_1^{b_k}(q_{i+1}(x_1 u_1^{1/(b_{k+1}-b_k)}) - q_i(x_1 u_1^{1/(b_{k+1}-b_k)}))) \\
&\qquad\qquad \times u_1^{1/(b_{k+1}-b_k)}\theta_{i,k}(x_1,u_1,\tau,\varepsilon)\,du_1\,dx_1,
\end{aligned}
$$

where $\theta_{i,k}$ (depending on f and g but not on φ) is a nonvanishing smooth function of x_1 and u_1 (in the domain of integration) for every τ and ε. The first pole of this integral will be a double pole if and only if

$$
\begin{aligned}
(A_m + a_mB_m)\tau + (C_m + a_mD_m)\varepsilon + a_m &= (B_m\tau + D_m\varepsilon) \\
&\quad + \frac{1}{b_{k+1} - b_k}[(A_m + a_mB_m)\tau + (C_m + a_mD_m)\varepsilon + a_m + 1] = -1.
\end{aligned}
$$

Keeping in mind the facts that

$$
\begin{aligned}
b_{k+1} - b_k + a_m &= b_{k+1} + a = a_{m+1}(\eta_i) \text{ and} \\
A_m + a_{m+1}B_m &= A_{m+1} + a_{m+1}B_{m+1},
\end{aligned}
$$

we obtain the following necessary and sufficient condition for the integral to have a double pole at $-\delta_0(g,f,\varepsilon)$:

$$
\begin{aligned}
\delta_0(g,f,\varepsilon) &= \frac{a_m + 1 + \varepsilon(C_m + a_mD_m)}{A_m + a_mB_m} \\
&= \frac{a_{m+1} + 1 + (C_{m+1} + a_{m+1}D_{m+1})\varepsilon}{A_{m+1} + a_{m+1}B_{m+1}},
\end{aligned}
$$

or in other words

$$
\delta_0(g,f,\varepsilon) = \delta_m\left(1 + \frac{\varepsilon}{\tilde{\delta}_m}\right) = \delta_{m+1}\left(1 + \frac{\varepsilon}{\tilde{\delta}_{m+1}}\right).
$$

The integrals $I^{i,k}_{\pm,0}(\tau,\varphi)$ for $k = K$ and $k = 0$ are simpler to handle. We only need one change of variable (respectively $\{x = x_1,\ u = u_1 x_1^{b_K}\}$ and $\{x = x_1 u_1^{1/b_1},\ u = u_1\}$) to reduce the integral to standard form. In particular, we record for our later analysis that

$$(5.9)\quad I^{i,K}_{\pm,0}(\tau,\varphi) = \int_{0<x<r,\ 0<u_1<1} u_1^{M_i\tau+N_i\varepsilon} x^{(A_M+a_MB_M)\tau+(C_M+a_MD_M)\varepsilon+a_M} \times \theta_{i,K}(x,u,\tau,\varepsilon)\varphi(x,q_i(x)+u_1x^{b_K}(q_{i+1}(x)-q_i(x)))\,du_1\,dx,$$

where M is such that $a_M(\eta_i) = b_k + a$. The details are left to the interested reader.

This completes the proof of Lemma 4 and also of Theorem 3.1. □

6. AN EXAMPLE

We recall that for a two-dimensional unweighted oscillatory integral with degenerate phase, δ_0 is always < 1. Therefore part (a) of Theorem 3.1 completely specifies the oscillation index of a two-dimensional unweighted oscillatory integral. However, it is not difficult to construct examples of weighted oscillatory integrals where f changes sign, $\delta_0(g,f,\varepsilon)$ is an odd integer and a simple pole of the integral in (3.1)–a situation not covered by the statement of the theorem. The following example shows that in general the oscillation index may be strictly larger than $\delta_0(g,f,\varepsilon)$ for such problems.

Let $f(x,y) = xy$, $g(x,y) = y^2$, $\varepsilon = 1$. Then,

$$\begin{aligned} I_+(\tau,\varphi) &= \int_{\{x>0,\,y>0\}\cup\{x<0,y<0\}} |f(x,y)|^\tau\,|g(x,y)|^\varepsilon\varphi(x,y)\,dy\,dx \\ &= \int_{x>0,\,y>0} x^\tau y^{\tau+2}(\varphi(x,y)+\varphi(-x,-y))\,dy\,dx, \end{aligned}$$

and

$$I_-(\tau,\varphi) = \int_{x>0,\,y>0} x^\tau y^{\tau+2}(\varphi(x,-y)+\varphi(-x,y))\,dy\,dx.$$

Note that $\tau_1 = 1$, and

$$a^+_{1,1} = a^-_{1,1} = \int_{y>0} y\,(\varphi(0,y)+\varphi(0,-y))\,dy.$$

Observe that even though $a^+_{1,1}$ and $a^-_{1,1}$ are nonzero numbers for generic φ,

$$(a^+_{1,1}+a^-_{1,1})\cos\left(-\frac{\pi}{2}\tau_1\right) + i(a^+_{1,1}-a^-_{1,1})\sin\left(-\frac{\pi}{2}\tau_1\right) = 0,$$

and hence the integral $I(\lambda,\varphi)$ decays faster than λ^{-1} for any φ. We therefore need to consider the next pole $\tau_2 = 2$. we write the analytic continuation of $I_\pm(\tau,\varphi)$ on $\tau > -2$ using integration by parts:

$$I_\pm(\tau,\varphi) = \int_{x,y>0} \frac{x^{\tau+1}}{\tau+1} y^{\tau+2} \frac{d}{dx}[\varphi(x,\pm y) + \varphi(-x,\mp y)]\, dy\, dx.$$

It is now clear that the pole of these functions at $\tau = -\tau_2 = -2$ is simple and the residues are given by the following integrals:

$$\begin{aligned} a^+_{2,1} &= \int_{y>0} y^{-2+2} \frac{d}{dx}[\varphi(x,y) + \varphi(-x,-y)]\Big|_{x=0} dy \\ &= \int_{y>0} (\varphi_x(0,y) - \varphi_x(0,-y))\, dy, \text{ and} \\ a^-_{2,1} &= \int_{y>0} y^{-2+2} \frac{d}{dx}[\varphi(x,-y) + \varphi(-x,y)]\Big|_{x=0} dy \\ &= \int_{y>0} (\varphi_x(0,-y) - \varphi_x(0,y))\, dy. \end{aligned}$$

In this case, τ_2 is an even integer and $a^+_{2,1} = -a^-_{2,1}$, so the coefficient of λ^{-2} in the asymptotic expansion of I (using Lemma 4.1) is

$$(a^+_{2,1} + a^-_{2,1}) \cos\left(\frac{\pi}{2}\tau_2\right) + i(a^+_{2,1} - a^-_{2,1}) \sin\left(\frac{\pi}{2}\tau_2\right) = 0.$$

Thus λ^{-2} is not the optimal bound either. We therefore have to proceed to $\tau_3 = 3$. Note that

$$\begin{aligned} I_+(\tau,\varphi) &\approx 2\int_{x>0,y>0} x^\tau y^{\tau+2}\Big[\varphi(0,0) + \frac{x^2}{2!}\varphi_{xx}(0,0) + \frac{y^2}{2!}\varphi_{yy}(0,0) \\ &\qquad + \frac{xy}{2!}\varphi_{xy}(0,0) + \cdots\Big]\, dy\, dx \\ &\approx 2\frac{\varphi(0,0)}{(\tau+1)(\tau+3)} + \frac{2}{2!}\frac{\varphi_{xx}(0,0)}{(\tau+3)^2} + \cdots, \end{aligned}$$

and

$$\begin{aligned} I_-(\tau,\varphi) &\approx 2\int_{x>0,y>0} x^\tau y^{\tau+2}\Big[\varphi(0,0) + \frac{x^2}{2!}\varphi_{xx}(0,0) + \frac{y^2}{2!}\varphi_{yy}(0,0) \\ &\qquad - \frac{xy}{2!}\varphi_{xy}(0,0) + \cdots\Big]\, dy\, dx \\ &\approx 2\frac{\varphi(0,0)}{(\tau+1)(\tau+3)} + \frac{2}{2!}\frac{\varphi_{xx}(0,0)}{(\tau+3)^2} + \cdots. \end{aligned}$$

Thus $\tau = -3$ is a double pole of I_+ and I_-, and $a^+_{3,2} = a^-_{3,2} = \varphi_{xx}(0,0) \neq 0$ in general. The calculations shown in the proof of Theorem 3.1(c) then yield

$$\beta(g, f, 1) = -3.$$

□

7. STATEMENT AND PROOF OF THEOREM 7.1

In the remainder of the paper we will deal with weighted oscillatory integrals for which the hypotheses of Theorem 3.1 fail and the oscillation index may potentially be smaller than $-\delta_0(g, f, \varepsilon)$. From the proof of Theorem 3.1, we have observed that the poles of $I^i_\pm(\tau, \varphi)$ are elements of finitely many arithmetic progressions. Let $\mathbb{P}$ denote the set of all poles of $I_\pm$:

$$\mathbb{P} := \{\rho \mid \text{there exists } \varphi \text{ such that } I_\pm(\tau, \varphi) \text{ has a pole at } \tau = -\rho\}.$$

Since it is difficult to keep track of the ordering of the elements in $\mathbb{P}$, we make a slight change of notation at this point. For $\rho \in \mathbb{P}$, let $a^\pm(\rho, k, \varphi)$ denote the coefficient of $(\tau + \rho)^{-k}$ in the Laurent series expansion of $I_\pm(\tau, \varphi)$ in a neighborhood of $\tau = \rho$, and let $b(\rho, k, \varphi)$ be the coefficient of $\lambda^{-\rho}(\ln \lambda)^{k-1}$ in the asymptotic expansion of $I(\lambda, \varphi)$.

Observe that, while the poles of $I_\pm(\tau, \varphi)$ are contained in $\mathbb{P}$ for a given φ, not all the elements of $\mathbb{P}$ make nontrivial contributions to the asymptotic expansion of $I(\lambda, \varphi)$. The reasons for this are several:

- Since the numbers $a^\pm(\rho, k, \varphi)$ depend heavily on φ, therefore for a specific choice of ρ and φ, $I^i_\pm(\tau, \varphi)$ may not have a pole at $\tau = -\rho$ for any i. This of course trivially implies $a^\pm(\rho, k, \varphi) = 0$. An example of this phenomenon is most easily seen by taking φ to be sufficiently degenerate near the origin, which has the effect that $I_\pm(\tau, \varphi)$ does not have a pole at $\tau = -\rho$ unless ρ is sufficiently large.
- Even if there exists an i such that $I^i_\pm(\tau, \varphi)$ has a pole at $\tau = -\rho$, the coefficients of the singular parts of $I^i_\pm(\tau, \varphi)$ at $\tau = -\rho$, when summed over i, may cancel out and give $a^+(\rho, k, \varphi) = a^-(\rho, k, \varphi) = 0$ for $k = 1, 2$. This would mean that $-\rho$ is not a pole of $I_\pm(\tau, \varphi)$, which would translate to $b(\rho, k, \varphi) = 0$. A simple example of this can be seen by redoing the example in the previous section with $f(x, y) = x^2 y$, $g(x, y) = xy^2$ and considering $\tau = -\frac{3}{2}$.
- Even when $I_\pm(\tau, \varphi)$ has a nontrivial singular part at $\tau = -\rho$, the contribution of this pole to the asymptotic expansion of $I(\lambda, \varphi)$ may be zero because of special properties like $\rho \in \mathbb{N}$ and $a^+(\rho, k, \varphi) = \pm a^-(\rho, k, \varphi)$, as we have seen in the example.

Let us therefore describe the set of relevant ρ's,

$$\mathcal{D} := \left\{\rho \in \mathbb{P} \mid \text{ there exists } \varphi \text{ such that } |b(\rho, 1, \varphi)|^2 + |b(\rho, 2, \varphi)|^2 \neq 0\right\}.$$

The oscillation index is then given by $\beta(g, f, \varepsilon) = -\min \mathcal{D}$. For a pair (f, g) whose Puiseux factorizations are known explicitly, one can compute using Lemma 4.1 the precise values of $a^{\pm}(\rho, k, \varphi)$ and $b(\rho, k, \varphi)$ for a generic φ, and hence obtain the oscillation index. However, for general f and g we cannot determine using properties of the Newton diagram alone the value of ρ that will provide the oscillation index. This requires a description of the set of elements $\rho \in \mathbb{P}$ which do not contribute to $I(\lambda, \varphi)$, and this in turn involves an understanding of the cancellation properties mentioned earlier. It is not clear whether a characterization of these cancellation properties can be obtained in terms of computable quantities, as they sometimes involve coefficients of the Puiseux series of f and g–information that is not encoded in the Newton diagrams. Our goal therefore is to find a finite subset of $-\mathcal{D}$ which depends only on the Newton diagrams of f and g in the admissible coordinates and whose maximum will then give a lower bound, possibly nonoptimal, for the oscillation index.

We give below a list of sufficient conditions for ρ to be in $\mathcal{D}$. Our result involves a specific subset of $\mathbb{P}$, namely

$$\mathcal{P}_i := \left\{ \rho = \frac{1 + \varepsilon N_i}{M_i} + \frac{j}{M_i} \mid j \geq 0 \right\}.$$

Recall that i ranges over the set of all real roots of f, and that M_i and N_i denote the multiplicity of the i-th element of this collection as a root of f and g respectively.

Theorem 7.1. *Let ρ_i be an element of $\mathcal{P}_i$ such that*

$$\rho_i = \begin{cases} \dfrac{1 + N_i\varepsilon}{M_i} & \text{if } M_i \text{ is even,} \\[2mm] \dfrac{1 + N_i\varepsilon}{M_i} & \text{if } M_i \text{ is odd and } \dfrac{1 + N_i\varepsilon}{M_i} \text{ is not an odd integer,} \\[2mm] \dfrac{2 + N_i\varepsilon}{M_i} & \text{if } M_i \text{ is odd, } M_i \geq 3, \text{ and } \dfrac{1 + N_i\varepsilon}{M_i} \text{ is an odd integer.} \end{cases}$$

Suppose the set $\mathcal{D}_0$ of all ρ_i-s defined as above is non-empty. Then

$$-\min \mathcal{D}_0 \leq \beta(g, f, \varepsilon) \leq -\delta_0(g, f, \varepsilon).$$

Remark. Note that in terms of the Newton diagram, we can read out the numbers M_i and N_i if for some admissible transformation there exist horizontal or vertical segments in the Newton diagrams of f and g whose heights from the x-axis are M_i and N_i respectively.

Proof of Theorem 7.1. In parts (a), (b) and (c) below, we treat the three cases mentioned in the statement of the theorem.

(a). Suppose M_i is even and $\rho = (1 + N_i\varepsilon)/M_i$. It is conceivable that $\tau = -\rho$ appears as a pole of $I^{\alpha,k}_{\pm,\cdot}(\tau,\varphi)$ for many k-s and α-s. For the moment let us assume that, for every $\alpha \in \mathcal{I}_+ \cup \mathcal{I}_-$, the order of multiplicity of $\tau = -\rho$ as a pole of $I^{\alpha}_{\pm}(\tau,\varphi)$ is at most 1. Since M_i is even, f does not change sign across the root q_i. Suppose that $f \geq 0$ on $\mathcal{R}_{i-1}$ and $\mathcal{R}_i$. Then both $i-1, i \in \mathcal{I}_+$.

It is easy to identify some integrals $I^{\alpha,k}_{\pm,\cdot}(\tau,\varphi)$ which have $\tau = -\rho$ as a pole. Certainly $I^{i,K}_{+,0}(\tau,\varphi)$ and $I^{i-1,K'}_{+,1}(\tau,\varphi)$ satisfy this condition, where K' is the value of K for $I^{i-1}_{+,1}$. The residue of $I^{i,K}_{+,0}(\tau,\varphi)$ at $\tau = -\rho$ is given by

$$\tag{7.1} \int_{0<x<r} x^{-(A_M + a_M B_M)\rho + (C_M + a_M D_M)\varepsilon + a_M} \theta_{i,K}(x, 0, -\rho, \varepsilon) \times \varphi(x, q_i(x))\,dx.$$

One has a similar expression for the residue of $I^{i-1,K'}_{+,1}(\tau,\varphi)$ at $\tau = -\rho$:

$$\tag{7.2} \int_{0<x<r} x^{-(A_{M'} + a_{M'} B_{M'})\rho + (C_{M'} + a_{M'} D_{M'})\varepsilon + a_{M'}} \tilde{\theta}_{i-1,K'}(x, 0, -\rho, \varepsilon) \times \varphi(x, q_i(x))\,dx.$$

Suppose first that (i,K) and $(i-1,K')$ are the only values of (α,k) that yield a pole at $\tau = -\rho$. Then by choosing φ such that $\varphi(x, q_i(x))$ is nonnegative and vanishes to a suitably high power of x at $x = 0$, we can ensure that the integrals in (7.1) converge and are indeed strictly positive. Since we have assumed that $\tau = -\rho$ does not appear as a pole of the meromorphic continuation of any integral other than $I^{i,K}_{+,0}(\tau,\varphi)$ and $I^{i-1,K'}_{+,1}(\tau,\varphi)$, the above observation ensures that $a^+(\rho,1) > 0$, $a^-(\rho,1) = 0$. This implies

$$\tag{7.3} (a^+(\rho,1) + a^-(\rho,1))\cos\left(\frac{\pi}{2}\rho\right) + i(a^+(\rho,1) - a^-(\rho,1))\sin\left(\frac{\pi}{2}\rho\right) \neq 0,$$

and we are done. Suppose next that there exists k different from K, K' such that $I^{i,k}_{+,0}(\tau,\varphi)$ or $I^{i-1,k}_{+,1}(\tau,\varphi)$ has $\tau = -\rho$ as a pole. From the expression (5.8) we see that the residue for such an integral would be a multiple of $\varphi(0,0)$. On the other hand, if there exist $\alpha \neq i, i-1$ such that $-\rho$ is a pole of $I^{\alpha}_{\pm}(\tau,\varphi)$, then the corresponding residue would be of one of the following three forms:

$$\begin{cases} \displaystyle\sum_{\mathbf{j}=(j_1,j_2,j_3)} c_{\mathbf{j}}(\alpha)\partial_1^{j_1}\partial_2^{j_2+j_3}\varphi(0,0), \\ \displaystyle\sum_j \int_{0<x<r} \Phi_{j,\alpha}(x,\rho,\varepsilon)[\partial_2^j\varphi](x, q_\alpha(x))\,dx, \text{ or} \\ \displaystyle\sum_j \int_{0<x<r} \tilde{\Phi}_{j,\alpha}(x,\rho,\varepsilon)[\partial_2^j\varphi](x, q_{\alpha+1}(x))\,dx, \end{cases}$$

where each of the sums is over a finite set of indices. Here $c_{\mathbf{j}}(\alpha)$-s are scalars, $\Phi_{j,\alpha}$ and $\tilde{\Phi}_{j,\alpha}$ are functions, and ∂_1 and ∂_2 denote derivatives with respect to the first and second argument respectively. The first case need not contribute to $a^{\pm}(\rho, 1)$ since we can find φ for which $\partial_1^{j_1}\partial_2^{j_2+j_3}\varphi(0,0) = 0$ for all $\mathbf{j}$, $|\mathbf{j}| \le M$ for some sufficiently large M. For the second and third cases, since q_α, $q_{\alpha+1} \neq q_i$ we can ensure the existence of a φ which satisfies all our previous hypotheses and moreover, $a^+(\rho, 1) \neq 0$, $a^-(\rho, 1) \neq 0$. This leads to (7.3).

Finally, suppose that there exists at least one $i_0 \in \mathcal{I}_+ \cup \mathcal{I}_-$ such that $\tau = -\rho$ is a double pole of $I_{\pm}^{i_0}(\tau, \varphi)$. Then the computation at the beginning of (c) in Section 5 (proof of Theorem 3.1) shows that at least one of the coefficients of $\lambda^{-\rho}$ or $\lambda^{-\rho}\ln\lambda$ in the asymptotic expansion of $I(\lambda, \varphi)$ is nonzero.

(b). If M_i is odd, let us assume without loss of generality that $i-1 \in \mathcal{I}_-$ and $i \in \mathcal{I}_+$. Using the argument in (a), we will reduce to and only treat the case when $-\rho$ is a simple pole that occurs only in $I_{+,0}^{i,K}(\tau, \varphi)$ and $I_{-,1}^{i-1,K'}(\tau, \varphi)$. Recalling the form of the residues given in (7.1) and (7.2), we can conclude by a genericity argument similar to the one given in (a) that $a^+(\rho, 1) > 0$ and $a^-(\rho, 1) > 0$, and hence $a^+(\rho, 1) + a^-(\rho, 1) > 0$. This means that if ρ is not an odd integer, then

$$(a^+(\rho, 1) + a^-(\rho, 1)) \cos\left(\frac{\pi}{2}\rho\right) \neq 0,$$

and (7.3) holds.

(c). Next suppose $\rho' = (1 + \varepsilon N_i)/M_i$ is an odd integer. Since $M_i \ge 3$, $\rho = \rho' + 1/M_i$ is not an integer. Furthermore, if we choose φ so that

$$\varphi(x, q_i(x)) \equiv 0, \quad \partial_2\varphi(x, q_i(x)) \not\equiv 0,$$

then the residues of $I_{+,0}^{i,K}(\tau, \varphi)$ and $I_{-,1}^{i-1,K'}(\tau, \varphi)$ at $\tau = -\rho$ are given by

$$a^+(\rho, 1) = \int_0^r x^{-(A_M + a_M B_M)\rho + (C_M + a_M D_M)\varepsilon + a_M + b_K} \times (q_{i+1}(x) - q_i(x))\theta_{i,K}(x, 0, -\rho, \varepsilon)[\partial_2\varphi](x, q_i(x))\, dx,$$

and

$$a^-(\rho, 1) = -\int_0^r x^{-(A_{M'} + a_{M'} B_{M'})\rho + (C_{M'} + a_{M'} D_{M'})\varepsilon + a_{M'} + b_{K'}} \times (q_i(x) - q_{i-1}(x))\tilde{\theta}_{i-1,K'}(x, 0, -\rho, \varepsilon)[\partial_2\varphi](x, q_i(x))\, dx$$

respectively. Note that, by the choice of a suitable φ, the two residues can be made nonzero but of opposite signs. We therefore conclude again from genericity considerations that $a^+(\rho,1) - a^-(\rho,1) > 0$. This implies that

$$(a^+(\rho,1) - a^-(\rho,1))\sin\left(\frac{\pi}{2}\rho\right) \neq 0,$$

which leads to (7.3). $\square$

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