

**OSCILLATORY INTEGRAL OPERATORS WITH
HOMOGENEOUS POLYNOMIAL PHASES IN SEVERAL
VARIABLES**

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ABSTRACT. We obtain L^2 decay estimates in λ for oscillatory integral operators T_λ whose phase functions are homogeneous polynomials of degree m and satisfy various genericity assumptions. The decay rates obtained are optimal in the case of $(2+2)$ -dimensions for any m , while in higher dimensions the result is sharp for m sufficiently large. The proof for large m follows from essentially algebraic considerations. For cubics in $(2+2)$ -dimensions, the proof involves decomposing the operator near the conic zero variety of the determinant of the Hessian of the phase function, using an elaboration of the general approach of Phong and Stein [10].

Keywords: Oscillatory integral operators, decay estimates, polynomial phase function, Newton polyhedron.

1. INTRODUCTION

Consider an oscillatory integral operator

$$(1.1) \quad T_\lambda f(x) = \int_{\mathbb{R}^{n_Z}} e^{i\lambda S(x,z)} a(x,z) f(z) dz, \quad x \in \mathbb{R}^{n_X},$$

where S is a real-valued phase function on $\mathbb{R}^{n_X} \times \mathbb{R}^{n_Z}$, $a \in C_0^\infty(\mathbb{R}^{n_X} \times \mathbb{R}^{n_Z})$ is a fixed amplitude supported in a compact neighborhood of the origin, and λ is a large parameter. For λ fixed, T_λ defines a bounded operator from $L^2(\mathbb{R}^{n_Z})$ to $L^2(\mathbb{R}^{n_X})$. We refer to this setting as “ $(n_X + n_Z)$ -dimensions”. A basic problem arising in many contexts [15],[14],[4] is determining the optimal rate of decay of the L^2 operator norm $\|T_\lambda\|$ as $\lambda \rightarrow \infty$. Typically, an upper bound for $\|T_\lambda\|$ is of the form

$$\|T_\lambda\| \leq C\lambda^{-r}(\log \lambda)^p, \quad \lambda \rightarrow \infty,$$

with $r > 0$ and $p \geq 0$ depend on S . For $n_X = n_Z = 1$, sharp results were obtained for C^ω phases by Phong and Stein [11], with the decay rate determined by the Newton polygon of $S(x,z)$. This was extended to most C^∞ phases by Rychkov [12], with the remaining cases settled by Greenblatt [2]. See also Seeger [13].

Extending all of these results to higher dimensions seems a difficult undertaking, and in the current work we focus on a more approachable problem, namely finding higher dimensional analogues of the results in Phong and Stein[9, 10] concerning homogeneous polynomials in $(1+1)$ -dimensions. One can assume that the phase function does not contain any monomial terms that are purely functions of x or of z , since these do not affect the L^2 operator norm, and then the main result of [10] is:

Theorem A (Phong and Stein). *Let $n_X = n_Z = 1$ and $S(x,z) = \sum_{j=1}^{m-1} a_j x^j z^{m-j}$. Assume that there exist $j \leq m/2$ and $k \geq m/2$ such that $a_j \neq 0$ and $a_k \neq 0$. Then*

$$\|T_\lambda\| \leq C\lambda^{-1/m}, \quad \lambda \rightarrow +\infty.$$

This result has been partially extended to $(2+1)$ -dimensions by Tang[18]. (See also Fu[1], where certain homogeneous polynomial phases, linear in one of the variables, are considered). The setup in [18] is as follows: write

$$S(x,z) = \sum_{j=1}^{m-1} P_j(x_1, x_2) z^{m-j},$$

where the P_j are homogeneous forms of degree j on \mathbb{R}^2 . Recall that a form P is nondegenerate if $\nabla P(x) \neq 0$ for $x \neq 0$; this is equivalent with P factoring over \mathbb{C} into $\deg(P)$ distinct linear factors. Let j_{\min} (respectively, j_{\max}) denote the first (respectively, last) index j for which P_j is not identically zero. The main result of [18] is:

Theorem B (Tang). *Let $S(x,z)$ be a homogeneous polynomial of degree m on $\mathbb{R}^2 \times \mathbb{R}$. Assume that $j_{\min} \leq 2m/3$, $j_{\max} \geq 2m/3$ and that both $P_{j_{\min}}, P_{j_{\max}}$ are nondegenerate on \mathbb{R}^2 . Then as $\lambda \rightarrow +\infty$,*

$$(1.2) \quad \|T_\lambda\| \leq \begin{cases} C\lambda^{-\frac{3}{2m}} & \text{if } m \geq 4 \\ C\lambda^{-\frac{1}{2}} \log(\lambda) & \text{if } m = 3 \\ C\lambda^{-\frac{1}{2}} & \text{if } m = 2. \end{cases}$$

These results are sharp, with the possible exception of $m = 3$, for which the lower bound $c\lambda^{-1/2}(\log \lambda)^{1/2}$ is known.

The purpose of the present work is to begin to deal with the difficulties encountered when trying to obtain versions of Theorem A and Theorem B in $(n_X + n_Z)$ -dimensions. Note that the hypotheses in those theorems are *generic*, i.e., they are satisfied by phase functions S belonging to an open, dense subset of the space of all homogeneous polynomials of given degree m . The emphasis of the present paper is on obtaining optimal decay rates for generic homogeneous phases in higher dimensions. We succeed in doing this in $(2 + 2)$ -dimensions, which we hope illuminates some of what needs to be done in higher dimensions as well. We will see that there is “low-hanging fruit”, namely phases of sufficiently high degree, where the optimal estimates for generic phases hold for essentially algebraic reasons.

In order to formulate the results, one needs to know the optimal possible decay rate for $\|T_\lambda\|$, given n_X , n_Z and m . Throughout the paper we assume that $n_X \geq n_Z$; it is of course always possible to ensure this, by taking adjoints if necessary. If $m = 2$, then the mixed Hessian matrix S''_{xz} is constant. Generically, $\text{rank}(S''_{xz}) = n_Z$ and it follows from the more general result of Hörmander[7] that $\|T_\lambda\| \leq C\lambda^{-\frac{1}{2}n_Z}$. For $m \geq 3$, the entries in S''_{xz} , being homogeneous of degree $m - 2$, must all vanish at the origin and in this case we prove the following:

Theorem 1.1. *Suppose $S(x, z)$ is homogeneous of degree $m \geq 3$ on $\mathbb{R}^{n_X} \times \mathbb{R}^{n_Z}$. Assume that it satisfies the Hörmander condition away from the origin:*

$$(1.3) \quad \text{rank}(S''_{xz}(x, z)) = n_Z \text{ for all } (x, z) \neq (0, 0).$$

Then

$$(1.4) \quad \|T_\lambda\| \leq \begin{cases} C\lambda^{-(n_X+n_Z)/(2m)} & \text{if } m > (n_X + n_Z)/n_Z \\ C\lambda^{-\frac{n_Z}{2}} \log(\lambda) & \text{if } m = (n_X + n_Z)/n_Z \\ C\lambda^{-\frac{n_Z}{2}} & \text{if } 2 \leq m < (n_X + n_Z)/n_Z. \end{cases}$$

Remark : For given n_X , n_Z and m , there may in fact be no phases satisfying (1.3). For example, if $n_X = n_Z = m$, then $\det(S''_{xz})$ is homogeneous of degree $n(m - 2)$. If this is odd, then $\det(S''_{xz})$ must have zeros away from $(0, 0)$.

Now, if $\min(n_X, n_Z) = n_Z \geq 2$ (which was not the case in [10] and [18]), the first estimate in (1.4) can be obtained relatively easily for phases that are (i) generic and (ii) of high degree, namely $m \geq n_X + n_Z$. In fact, generic phases can be shown to satisfy a *rank one* condition, which, while relatively weak, allows one to obtain the optimal decay rate for large m .

Definition. *A homogeneous phase function $S(x, z)$ is said to satisfy the rank one condition if*

$$(1.5) \quad \text{rank}(S''_{xz}(x, z)) \geq 1 \text{ for all } (x, z) \neq (0, 0),$$

i.e., if S''_{xz} has at least one nonzero entry at every point in $\mathbb{R}^{n_X+n_Z} \setminus (0, 0)$.

If $n_Z = 1$, then $S''_{xz} = (S''_{x_1z}, \dots, S''_{x_{n_X}z})$ consists of n_X polynomials, each homogeneous of degree $m - 2$ on \mathbb{R}^{n_X+1} , and in general they may have a common zero on $\mathbb{R}^{n_X+1} \setminus (0, 0)$. The decompositions of T_λ in [10] ($n_X = 1$) and [18] ($n_X = 2$) were adapted to the geometry of these zeros. However, for $n_Z \geq 2$, one can show that these common zeros are generically not present. (The precise definition of genericity will be described in §3.)

Proposition 1.2. *If $n_X \geq n_Z \geq 2$, a generic homogeneous polynomial phase function $S(x, z)$ on $\mathbb{R}^{n_X+n_Z}$ satisfies the rank one condition (1.5).*

For $m \geq n_X + n_Z$, the optimal decay rate from (1.4) is $\leq 1/2$, which allows us to use the $(1+1)$ -dimensional operator Van der Corput lemma of [10] to obtain:

Theorem 1.3. *For a homogeneous phase function $S(x, z)$ of degree m satisfying the rank one condition (1.5) on $\mathbb{R}^{n_X+n_Z}$,*

$$(1.6) \quad \|T_\lambda\| \leq \begin{cases} C\lambda^{-(n_X+n_Z)/(2m)} & \text{if } m > n_X + n_Z, \\ C\lambda^{-1/2} \log \lambda & \text{if } m = n_X + n_Z, \\ C\lambda^{-1/2} & \text{if } 2 \leq m < n_X + n_Z. \end{cases}.$$

Thus, for generic phases and $n_Z \geq 2$, the true analytic difficulties lie in the range $3 \leq m < n_X + n_Z$. In particular, to obtain the full picture for generic phases in $2+2$ dimensions, it remains only to analyze the case for generic cubics. Here ‘‘generic’’ will mean that the hypotheses of Thm. 1.4 below are satisfied. In §4 we will show that these hold for an explicit open, dense subset of the space of cubics.

If $S(x, z)$ is a homogeneous cubic on \mathbb{R}^{2+2} , the entries of the Hessian matrix

$$(1.7) \quad S''_{xz}(x, z) = \begin{bmatrix} S''_{x_1 z_1} & S''_{x_1 z_2} \\ S''_{x_2 z_1} & S''_{x_2 z_2} \end{bmatrix}$$

are linear forms on \mathbb{R}^4 , and $\Phi(x, z) = \det(S''_{xz}(x, z))$ is a quadratic form,

$$(1.8) \quad \Phi(x, z) = \frac{1}{2}x^t P x + x^t Q z + \frac{1}{2}z^t R z,$$

where P, Q and R are 2×2 matrices with P and R symmetric. Let $\text{Res}[f, g]$ denote the resultant of two homogeneous polynomials in two variables, so that f and g share a common zero in $\mathbb{C}^2 \setminus 0$ iff $\text{Res}[f, g] = 0$; Res will be discussed in more detail in §3 below. We may now state the main result of this paper.

Theorem 1.4. *Assume that $S(x, z)$ is a homogeneous cubic phase function on \mathbb{R}^{2+2} with $\Phi(x, z) = \det(S''_{xz})$ given by (1.8) such that*

$$(1.9) \quad P \text{ and } R \text{ are nonsingular;}$$

$$(1.10) \quad P - QR^{-1}Q^t \text{ and } R - Q^t P^{-1}Q \text{ are nonsingular; and}$$

$$(1.11) \quad \left\{ \begin{array}{l} \text{Res}[x^t(P - QR^{-1}Q^t)x, x^t QR^{-1}(R - Q^t P^{-1}Q)R^{-1}Q^t x] \neq 0, \\ \text{Res}[z^t(R - Q^t P^{-1}Q)z, z^t Q^t P^{-1}(P - QR^{-1}Q^t)P^{-1}Qz] \neq 0. \end{array} \right\}$$

In addition, if both P and R are indefinite, assume

$$(1.12) \quad \left\{ \begin{array}{l} \text{Res}[x^t P x, x^t(P - QR^{-1}Q^t)x] (= -\text{Res}[x^t P x, x^t QR^{-1}Q^t x]) \neq 0, \\ \text{Res}[z^t R z, z^t(R - Q^t P^{-1}Q)z] (= -\text{Res}[z^t R z, z^t Q^t P^{-1}Qz]) \neq 0. \end{array} \right\}$$

Then $\|T_\lambda\| \leq C\lambda^{-2/3}$ as $\lambda \rightarrow \infty$.

Remarks.

- (1) In (1.11) and (1.12), $\text{Res}[f, g]$ is the resultant of two homogeneous polynomials in two variables, which vanishes iff f and g have a common zero in $\mathbb{C}^2 \setminus 0$ (cf. [17]). Basic facts concerning resultants will be reviewed in §3.

- (2) The hypotheses are certainly not necessary for the decay rate of $\lambda^{-2/3}$ to hold. See the discussion in §§4.2. However, determining exactly which phases have this optimal decay rate does not seem to be easy.
- (3) If (1.9) holds, then each matrix in (1.10) is nonsingular iff the other is, and this is equivalent with the quadratic form Φ being nondegenerate (cf. (5.3)).
- (4) The hypotheses have geometric interpretations which will be described in §5 and §6.
- (5) It is natural to ask whether the hypotheses imply that the natural projections $\tilde{\pi}_L : C_S = \{(x, d_x S(x, z); z, -d_z S(x, z))\} \rightarrow T^*\mathbb{R}_x^2$ and $\tilde{\pi}_R : C_S \rightarrow T^*\mathbb{R}_z^2$ belong to singularity classes, such as folds and cusps, for which the decay estimates are known [4]. At $(x, z) = (0, 0)$, both $d\tilde{\pi}_L$ and $d\tilde{\pi}_R$ drop rank by 2. The simplest C^∞ singularities of corank 2 are the umbilics [6], but the conditions in Thm. 1.4 do not seem to imply that $\tilde{\pi}_L$ and $\tilde{\pi}_R$ have these singularities.

2. NONDEGENERATE AND RANK ONE CASES

Proof of Theorem 1.1. Since the support of the amplitude in (1.1) is compact, we may assume that $|(x, z)| \leq 1$ on $\text{supp}(a)$. Let $\{\psi_k\}$ be a dyadic partition of unity, $\sum_{k=0}^{\infty} \psi_k(x, z) \equiv 1$, satisfying

$$(2.1) \quad \text{supp}(\psi_k) \subseteq \{2^{-k-1} \leq |(x, z)| \leq 2^{-k+1}\}, \quad \|\partial_x^\alpha \partial_z^\beta \psi_k\|_\infty \leq C_{\alpha\beta} 2^{(|\alpha|+|\beta|)k}.$$

Set $a_k = \psi_k a$ and let $T_\lambda^k f(x) = \int e^{i\lambda S(x, z)} a_k(x, z) f(z) dz$, so that $T_\lambda = \sum_{k=0}^{\infty} T_\lambda^k$. By the nondegeneracy hypothesis (1.3), for each $(x_0, z_0) \neq (0, 0)$, there is a nonsingular $n_Z \times n_Z$ minor of $S''_{x'z}(x_0, z_0)$. Since the entries in $S''_{x'z}$ are all homogeneous of degree $m-2$, the same minor is nonsingular for all (x, z) in a conic neighborhood \mathcal{U} of (x_0, z_0) . A finite number of such neighborhoods cover $\mathbb{R}^{n_X+n_Z} \setminus (0, 0)$, and so we can assume that $\text{supp}(a) \subset \mathcal{U}$. Furthermore, by a linear change of variable, we may assume that $\det(S''_{x'z}) \neq 0$ on \mathcal{U} , where $x = (x', x'') \in \mathbb{R}^{n_Z} \times \mathbb{R}^{n_X-n_Z}$.

Now, as in [10], we can estimate $\|T_\lambda^k\|$ in two ways. First, we observe that the x and z supports of the a_k have measures $\leq C2^{-n_X k}$ and $C2^{-n_Z k}$ respectively, so an application of Young's inequality gives

$$(2.2) \quad \|T_\lambda^k\| \leq C2^{-\frac{n_X+n_Z}{2}k}.$$

Secondly, on $\{1/2 \leq |(x, z)| \leq 2\}$, the lower bound $|\det(S''_{x'z})| \geq c > 0$ implies $\|(S''_{x'z})^{-1}\| \leq C' < \infty$. By homogeneity, we have $\|(S''_{x'z})^{-1}\| \leq C'2^{(m-2)k}$ on $\text{supp}(a_k)$. The standard proof of Hörmander's estimate for nondegenerate oscillatory integral operators (e.g., [3, Lem. 2.3]) then shows that, for fixed x'' , the operator norm of $f(\cdot) \mapsto T_\lambda^k f(\cdot, x'')$ is $\leq C(2^{-(m-2)k} \lambda)^{-n_X/2}$. Combining this with the size of the support in x'' , we obtain

$$(2.3) \quad \begin{aligned} \|T_\lambda^k\| &\leq C(2^{-(m-2)k} \lambda)^{-n_Z/2} (2^{-k})^{\frac{n_X-n_Z}{2}} \\ &\leq C2^{((m-2)n_Z-n_X+n_Z)k/2} \lambda^{-n_Z/2}. \end{aligned}$$

The estimates in (2.2) and (2.3) are comparable if and only if

$$2^{-(n_X+n_Z)k/2} \sim 2^{((m-2)n_Z-n_X+n_Z)k/2} \lambda^{-n_Z/2}, \quad \text{or } 2^k \sim \lambda^{1/m}.$$

For $0 \leq k \leq m^{-1} \log_2 \lambda$, (2.3) is smaller, while for $k > m^{-1} \log_2 \lambda$, (2.2) is smaller. Thus

$$\begin{aligned} \|T_\lambda\| &\leq \sum_{k=0}^{\infty} \|T_\lambda^k\| \\ &\leq C \left[\lambda^{-n_Z/2} \sum_{k=0}^{\frac{1}{m} \log_2 \lambda} 2^{((m-2)n_Z - n_X + n_Z)k/2} + \sum_{k=\frac{1}{m} \log_2 \lambda}^{\infty} 2^{-(n_X + n_Z)k/2} \right]. \end{aligned}$$

If $m > (n_X + n_Z)/n_Z$, then $(m-2)n_Z - n_X + n_Z > 0$, and the first sum is $\lesssim \lambda^{-n_Z/2} \lambda^{((m-2)n_Z - n_X + n_Z)/(2m)} = C \lambda^{-(n_X + n_Z)/(2m)}$. If $m = (n_X + n_Z)/n_Z$, then the first sum is $\lesssim \lambda^{-n_Z/2} \log_2 \lambda$, while if $m < (n_X + n_Z)/2$, it is $\lesssim \lambda^{-n_Z/2}$. On the other hand, the second sum is $\lesssim \lambda^{-(n_X + n_Z)/(2m)}$ in all cases. This yields (1.4) and thus finishes the proof of Thm. 1.1. \square

Proof of Theorem 1.3. Under the rank one assumption, for each $(x_0, z_0) \neq (0, 0)$ there are indices i_0, j_0 with $1 \leq i_0 \leq n_X$, $1 \leq j_0 \leq n_Z$, such that $S''_{x_{i_0} z_{j_0}}(x_0, z_0) \neq 0$, and this holds on a conic neighborhood \mathcal{U} of (x_0, z_0) . As above, a finite number of such \mathcal{U} cover $\mathbb{R}^{n_X + n_Z} \setminus (0, 0)$, and we may assume $a(x, z)$ is supported on one such \mathcal{U} . By linear changes of variables, we may then assume that $i_0 = j_0 = 1$. Writing $x = (x_1, x')$ and $z = (z_1, z')$, we argue as above, this time applying the nondegenerate estimate in the x_1, z_1 variables only. We thus obtain, in place of (2.3), the estimate

$$(2.4) \quad \|T_\lambda^k\| \lesssim (2^{(m-2)k} \lambda)^{-1/2} 2^{-(n_X + n_Z - 2)k/2} \lesssim \lambda^{-1/2} 2^{(m - n_X - n_Z)k/2},$$

while (2.2) applies as before. These two estimates for $\|T_\lambda^k\|$ are comparable if and only if

$$2^{(m - n_X - n_Z)k/2} 2^{(n_X + n_Z)k/2} \sim \lambda^{1/2}, \text{ i.e., if and only if } 2^k \sim \lambda^{1/m},$$

with (2.4) smaller if $0 \leq k \leq (1/m) \log_2 \lambda$ and (2.2) smaller if $k > (1/m) \log_2 \lambda$. This leads to the estimate

$$\begin{aligned} \|T_\lambda\| &\lesssim \lambda^{-1/2} \sum_{k=0}^{\frac{1}{m} \log_2 \lambda} 2^{(m - n_X - n_Z)k/2} + \sum_{k=\frac{1}{m} \log_2 \lambda}^{\infty} 2^{-(n_X + n_Z)k/2} \\ &\lesssim \begin{cases} \lambda^{-(n_X + n_Z)/(2m)} & \text{for } m > n_X + n_Z \\ \lambda^{-1/2} \log_2 \lambda & \text{for } m = n_X + n_Z, \\ \lambda^{-1/2} & \text{for } m < n_X + n_Z, \end{cases} \end{aligned}$$

proving Thm. 1.3. \square

Remark. It follows from their proofs that both Thm. 1.1 and Thm. 1.3 have conically localized variants. Rather than belonging to C_0^∞ , the amplitude $a(x, z)$ is assumed to be of compact support in $C^\infty(\mathbb{R}^{n_X + n_Z} \setminus (0, 0))$, and homogeneous of degree zero (jointly in (x, z)) for $|(x, z)|$ sufficiently small. The phase function $S(x, z)$ is also only assumed to satisfy (1.3) or (1.5) on $\text{supp}(a) \setminus (0, 0)$. The key point is that $\psi_k \cdot a$ still satisfies (2.1). This observation will be used in the proof of Thm. 1.4 to reduce the argument to a small conic neighborhood of the critical variety.

3. GENERIC HOMOGENEOUS POLYNOMIAL PHASES

To understand why the rank one hypothesis of Thm. 1.3 holds for generic phase functions $S(x, z)$ of degree $m \geq n_X + n_Z$ in $(n_X + n_Z)$ -dimensions, $n_Z \geq 2$, as do the assumptions of Thm. 1.4 for generic cubics in $(2 + 2)$ -dimensions, consider the finite dimensional vector spaces of phase functions and their Hessians. For $m, N \in \mathbb{N}$, the space $S^m \mathbb{R}^N$ of homogeneous polynomials of degree m on \mathbb{R}^N is of dimension $\binom{m+N-1}{m}$ (see for example [16, p. 139]). When $\mathbb{R}^N = \mathbb{R}^{n_X} \times \mathbb{R}^{n_Z}$, we are only interested in polynomial phase functions which do not contain monomials that are functions of x or z alone, since these leave the L^2 operator norm unchanged. Thus, we define $\mathfrak{S}^m \mathbb{R}^{n_X+n_Z}$ as the subspace of $S^m(\mathbb{R}^{n_X+n_Z})$ consisting of such polynomials. Clearly,

$$(3.1) \quad \dim \mathfrak{S}^m \mathbb{R}^{n_X+n_Z} = \binom{m+n_X+n_Z-1}{m} - \binom{m+n_X-1}{m} - \binom{m+n_Z-1}{m}.$$

For $S(x, z) \in \mathfrak{S}^m \mathbb{R}^{n_X+n_Z}$, the mixed Hessian is

$$(3.2) \quad S''_{xz}(x, z) = \left(\frac{\partial^2 S(x, z)}{\partial x_i \partial z_j} \right)_{\substack{1 \leq i \leq n_X \\ 1 \leq j \leq n_Z}} \in \mathbb{M}_{n_X \times n_Z} [S^{m-2} \mathbb{R}^{n_X+n_Z}],$$

where the last space is the vector space of $n_X \times n_Z$ matrices with entries from $S^{m-2} \mathbb{R}^{n_X+n_Z}$. As mentioned earlier, if $m = 2$ then S''_{xz} is constant and $\|T_\lambda\| \lesssim \lambda^{-r}$, $r = \text{rank}(S''_{xz})/2$. Thus, we will always assume that $m \geq 3$. Now, in $(1 + 1)$ -dimensions, $\dim \mathfrak{S}^m \mathbb{R}^{1+1} = m - 1 = \dim \mathbb{M}_{1 \times 1} [S^{m-2} \mathbb{R}^{1+1}]$, and the Hessian map $S \mapsto \mathfrak{h}(S) = S''_{xz}$ is an isomorphism. However, for $n_X \geq 2$, $\dim \mathfrak{S}^m \mathbb{R}^{n_X+n_Z} < \dim \mathbb{M}_{n_X \times n_Z} [S^{m-2} \mathbb{R}^{n_X+n_Z}]$, and the range of \mathfrak{h} is of positive (typically very high) codimension. Note that by commutativity of mixed partial derivatives, we have

$$\begin{aligned} (S_{x_i z_j})_{x_{i'}} &= (S_{x_{i'} z_j})_{x_i}, \text{ for all } 1 \leq i < i' \leq n_X, 1 \leq j \leq n_Z, \text{ and} \\ (S_{x_i z_j})_{z_{j'}} &= (S_{x_i z_{j'}})_{z_j}, \text{ for all } 1 \leq i \leq n_X, 1 \leq j < j' \leq n_Z. \end{aligned}$$

In fact, these linear equations characterize the range of \mathfrak{h} :

Proposition 3.1. *Let $\mathbb{M}_{\mathfrak{h}} [S^{m-2} \mathbb{R}^{n_X+n_Z}] \leq \mathbb{M}_{n_X \times n_Z} [S^{m-2} \mathbb{R}^{n_X+n_Z}]$ be the subspace consisting of all $H(x, z) = (H_{ij}(x, z))$, $1 \leq i \leq n_X$, $1 \leq j \leq n_Z$, such that*

$$(3.3) \quad (H_{ij})_{x_{i'}} = (H_{i'j})_{x_i} \text{ for all } 1 \leq i < i' \leq n_X, \quad 1 \leq j \leq n_Z, \text{ and}$$

$$(3.4) \quad (H_{ij})_{z_{j'}} = (H_{ij'})_{z_j} \text{ for all } 1 \leq i \leq n_X, \quad 1 \leq j < j' \leq n_Z.$$

Then the Hessian map $\mathfrak{h}(S) := S''_{xz}$ is an isomorphism,

$$\mathfrak{h} : \mathfrak{S}^m \mathbb{R}^{n_X+n_Z} \rightarrow \mathbb{M}_{\mathfrak{h}} [S^{m-2} \mathbb{R}^{n_X+n_Z}].$$

Proof. We first show that \mathfrak{h} is injective. Write $S(x, z) = \sum c_{\alpha\beta} x^\alpha z^\beta$, where α, β vary over the index set $\{|\alpha| + |\beta| = m, |\alpha|, |\beta| > 0\}$. Then

$$\mathfrak{h}(S)_{ij}(x, z) = \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|, |\beta|>0}} \alpha_i \beta_j a_{\alpha\beta} x^{\alpha - e_i} z^{\beta - \bar{e}_j},$$

where e_i and \bar{e}_j denote the standard basis elements of \mathbb{Z}^{n_X} and \mathbb{Z}^{n_Z} respectively. Thus, if $S \in \ker(\mathfrak{h})$, so that $\mathfrak{h}(S)_{ij} = 0 \in S^{m-2} \mathbb{R}^{n_X+n_Z}$, for all $1 \leq i \leq n_X$, $1 \leq j \leq n_Z$, then $\alpha_i \beta_j a_{\alpha\beta} = 0$, for all α, β, i, j . But for any α, β with $|\alpha|, |\beta| >$

0, there exist i and j with $\alpha_i \beta_j \neq 0$, so that $a_{\alpha\beta} = 0$, for all α, β , and hence $S = 0 \in \mathfrak{S}^m \mathbb{R}^{n_X + n_Z}$.

Next we prove that \mathfrak{h} is surjective. Let $H = (H_{ij}) \in \mathbb{M}_{\mathfrak{h}}[S^{m-2} \mathbb{R}^{n_X + n_Z}]$, and write $H_{ij}(x, z) = \sum_{|\alpha|+|\beta|=m-2} b_{\alpha\beta}^{ij} x^\alpha z^\beta$. For all $\alpha \in \mathbb{Z}_+^{n_X}$ and $\beta \in \mathbb{Z}_+^{n_Z}$ with $|\alpha| > 0$, $|\beta| > 0$ and $|\alpha| + |\beta| = m$, define

$$(3.5) \quad a_{\alpha\beta} = \frac{1}{\alpha_i \beta_j} b_{\alpha - e_i, \beta - \bar{e}_j}^{ij}$$

for any $i \in \{1, \dots, n_X\}$, and $j \in \{1, \dots, n_Z\}$ such that $\alpha_i \neq 0$ and $\beta_j \neq 0$. This is well-defined, because the right hand side of (3.5) is independent of the choice of i and j : by (3.3) and (3.4), we have $(H_{ij})_{x_i, z_j} = (H_{i'j'})_{x_i, z_j}$, so that

$$\sum_{\alpha, \beta} \alpha_{i'} \beta_{j'} b_{\alpha\beta}^{ij} x^{\alpha - e_{i'}} z^{\beta - \bar{e}_{j'}} = \sum_{\mu, \nu} \mu_i \nu_j b_{\mu\nu}^{i'j'} x^{\mu - e_i} z^{\nu - \bar{e}_j}.$$

Hence, if $\alpha - e_{i'} = \mu - e_i$ and $\beta - \bar{e}_{j'} = \nu - \bar{e}_j$, we have

$$(3.6) \quad \alpha_{i'} \beta_{j'} b_{\alpha\beta}^{ij} = \mu_i \nu_j b_{\mu\nu}^{i'j'}, \quad \text{or} \quad \frac{b_{\alpha\beta}^{ij}}{\mu_i \nu_j} = \frac{b_{\mu\nu}^{i'j'}}{\alpha_{i'} \beta_{j'}}.$$

First suppose $i \neq i'$ and $j \neq j'$. Then

$$\alpha_{i'} - 1 = \mu_{i'}, \quad \mu_i - 1 = \alpha_i, \quad \beta_{j'} - 1 = \nu_{j'}, \quad \nu_j - 1 = \beta_j,$$

and (3.6) translates to

$$\frac{1}{(\alpha_i + 1)(\beta_j + 1)} b_{\alpha\beta}^{ij} = \frac{1}{\alpha_{i'} \beta_{j'}} b_{\alpha + e_i - e_{i'}, \beta + \bar{e}_j - \bar{e}_{j'}}^{i'j'}.$$

Replacing α by $\alpha - e_i$ and β by $\beta - \bar{e}_j$ we obtain the desired conclusion,

$$\frac{1}{\alpha_i \beta_j} b_{\alpha - e_i, \beta - \bar{e}_j}^{ij} = \frac{1}{\alpha_{i'} \beta_{j'}} b_{\alpha - e_{i'}, \beta - \bar{e}_{j'}}^{i'j'}.$$

The cases $i = i', j \neq j'$ and $i \neq i', j = j'$ are similar and are left to the reader. Finally, it is an easy matter to check that

$$\text{if } S(x, z) = \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|, |\beta| > 0}} a_{\alpha\beta} x^\alpha z^\beta, \text{ then } (S)_{x_i z_j} = H_{ij} \text{ for all } 1 \leq i \leq n_X, 1 \leq j \leq n_Z,$$

which completes the proof. \square

We can now prove that generic phases satisfy the rank one condition.

Proof of Proposition 1.2. Since \mathfrak{h} is an isomorphism, to show that a property holds for generic $S \in \mathfrak{S}^m \mathbb{R}^{n_X + n_Z}$, it suffices to show that it holds for generic $H = (H_{ij}) \in \mathbb{M}_{\mathfrak{h}} = \mathbb{M}_{\mathfrak{h}}[S^{m-2} \mathbb{R}^{n_X + n_Z}]$. Thus, to prove Prop. 1.2, it suffices to show that if $n_X \geq n_Z \geq 2$, then a generic element of $\mathbb{M}_{\mathfrak{h}}$ satisfies the rank one condition. In turn, it suffices to find a subset $I \subset \{1, \dots, n_X\} \times \{1, \dots, n_Z\}$, $|I| = n_X + n_Z$ such that

$$\mathcal{U}_I = \left\{ H \in \mathbb{M}_{\mathfrak{h}} : \bigcap_{(i,j) \in I} \{(x, z) \in \mathbb{R}^{n_X + n_Z} : H_{ij}(x, z) = 0\} = \{0\} \right\}$$

is a Zariski open subset of $\mathbb{M}_{\mathfrak{h}}$.

To do this, as well as to explain conditions (1.11), (1.12) in Thm. 1.4, we make use of the multivariate resultant, which we briefly recall (see [17] for background material on resultants). There exists a polynomial $\text{Res}[f_1, \dots, f_N]$ in the variables

$\{c_\gamma^k : |\gamma| = d_k, 1 \leq k \leq N\}$ such that if $f_1(y), \dots, f_N(y)$ are N homogeneous polynomials of degree d_1, \dots, d_N on \mathbb{C}^N , $f_k(y) = \sum_{|\gamma|=d_k} c_\gamma^k y^\gamma$, then f_1, \dots, f_N have a common zero on $\mathbb{C}^N \setminus \{0\}$ if and only if $\text{Res}[f_1, \dots, f_N] = 0$. Hence, if $\text{Res}[f_1, \dots, f_N] \neq 0$, then f_1, \dots, f_N have no common zero on $\mathbb{C}^N \setminus \{0\}$, and thus on $\mathbb{R}^N \setminus \{0\}$. For each k , Res is a polynomial in the coefficients $(c_\gamma^k)_{|\gamma|=d_k}$ of degree $d_1 \cdots d_{k-1} d_{k+1} \cdots d_N$.

Applying this with $N = n_X + n_Z$, $y = (x, z)$, $d_k = m - 2$ for all k , and $f_k = H_{i_k j_k}$, where $I = \{(i_k, j_k) : 1 \leq k \leq N\}$, if we can find one element H^0 of $\mathbb{M}_\mathfrak{h}$ such that $\text{Res}[H_{i_1 j_1}^0, \dots, H_{i_N j_N}^0] \neq 0$, then

$$H \in \mathbb{M}_\mathfrak{h} \mapsto \text{Res}[H_{i_1 j_1}, \dots, H_{i_N j_N}]$$

is a polynomial of degree $(n_X + n_Z)(m - 2)^{n_X + n_Z - 1}$ in the coefficients of H which does not vanish identically. Hence

$$\mathcal{U}_I = \{H \in \mathbb{M}_\mathfrak{h} : \text{Res}[H_{i_1 j_1}, \dots, H_{i_N j_N}] \neq 0\}$$

is a Zariski open subset of $\mathbb{M}_\mathfrak{h}$, and for every $H \in \mathcal{U}_I$,

$$\bigcap_{1 \leq k \leq N} \{(x, z) : H_{i_k j_k}(x, z) = 0\} = (0, 0),$$

so that at every point of $\mathbb{R}^{n_X + n_Z} \setminus \{0\}$ at least one element of $(H_{ij}(x, z))$ is nonzero. Thus, a generic element of $\mathbb{M}_\mathfrak{h}$ satisfies the rank-one condition (1.5).

We construct such an H^0 first in the case of $n_X = n_Z = n$. Let

$$(3.7) \quad H^0(x, z) = \sum_{i=1}^n x_i^{m-2} e_{ii} + \sum_{i=2}^n z_i^{m-2} e_{i-1, i} + z_n^{m-2} e_{n1},$$

where $\{e_{ij}\}_{1 \leq i, j \leq n}$ is the standard basis of $\mathbb{M}_{n \times n}[\mathbb{R}]$. Then $H^0 \in \mathbb{M}_\mathfrak{h}$, since, in (3.3) and (3.4), all of the terms are zero. In fact, one easily sees that $H^0 = S''_{xz}$ for

$$(3.8) \quad S(x, z) = \frac{1}{m-1} \left(\sum_{i=1}^n x_i^{m-1} z_i + \sum_{i=2}^n x_i z_i^{m-1} + x_n z_1^{m-1} \right).$$

Letting $I = \{(i, i) : 1 \leq i \leq n\} \cup \{(i-1, i) : 2 \leq i \leq n\} \cup \{(n, 1)\}$, we have $\bigcap_{(i, j) \in I} \{(x, z) : H_{ij}^0(x, z) = 0\} = (0, 0)$, and $\mathcal{U}_I \subset \mathbb{M}_\mathfrak{h}$ is Zariski open. Hence, the rank one condition (1.5) holds for generic phase functions $S \in \mathfrak{S}^m \mathbb{R}^{n+n}$.

For the case $n_X > n_Z \geq 2$, we use the above construction in the $n_Z \times n_Z$ submatrix (H_{ij}) , $1 \leq i, j \leq n_Z$, with corresponding index set \tilde{I} , $|\tilde{I}| = 2n_Z$. We then place the monomials x_i^{m-2} , $n_Z + 1 \leq i \leq n_X$ in any $n_X - n_Z$ distinct entries \bar{I} of the $(n_X - n_Z) \times n_Z$ submatrix (H_{ij}) , $n_Z + 1 \leq i \leq n_X$, $1 \leq j \leq n_Z$. Then (3.3) and (3.4) are satisfied, and letting $I = \tilde{I} \cup \bar{I}$, we obtain $\bigcap_{(i, j) \in I} \{H_{ij}(x, z) = 0\} = \{0\}$. Thus, $\mathcal{U}_I \subset \mathbb{M}_\mathfrak{h}[S^{m-2} \mathbb{R}^{n_X + n_Z}]$ is Zariski open and so the rank one condition (1.5) holds for generic $S \in \mathfrak{S}^m \mathbb{R}^{n_X + n_Z}$. This finishes the proof of Prop. 1.2. \square

4. SHARPNESS AND RELATION WITH NEWTON DISTANCE

4.1. Optimality of decay rates.

Theorem 4.1. *If $S(x, z)$ is a real polynomial, homogeneous of degree m on $\mathbb{R}^{n_X+n_Z}$, and T_λ as defined by (1.1), then*

$$(4.1) \quad \|T_\lambda\| \geq c\lambda^{-(n_X+n_Z)/2m}, \lambda \rightarrow \infty.$$

If in addition, $n_X \geq n_Z$ and $S(x, z)$ satisfies (1.3) at some point (x_0, z_0) , then

$$(4.2) \quad \|T_\lambda\| \geq c\lambda^{-n_Z/2}, \lambda \rightarrow \infty.$$

Remark. Thus, Thm. 1.4 is sharp, as is Thm. 1.1 except possibly for the $\log(\lambda)$ term when $m = (n_X + n_Z)/n_Z$. Furthermore, Thm. 1.3 is sharp for $m \geq n_X + n_Z$, again except possibly for the $\log(\lambda)$ term when $m = n_X + n_Z$.

Proof. For (4.1), we adapt the argument of [10] from the $(1+1)$ -dimensional setting. Pick an $(x_0, z_0) \in \text{supp}(a)$ with $x_0 \neq 0$, $z_0 \neq 0$. Let $\epsilon > 0$ be small enough so that

$$\left| \arg(e^{iS(x,z)}) - \arg(e^{iS(x_0,z_0)}) \right| < \frac{\pi}{8}$$

for $x \in B(x_0, \epsilon)$ and $x \in B(z_0, \epsilon)$. Then we can find an $f \in C_0^\infty(B(z_0, \epsilon))$ with $\|f\|_{L^2} = 1$ and

$$|T_1 f(x)| = \left| \int e^{iS(x,z)} a(x,z) f(z) dz \right| \geq C > 0$$

for $x \in B(x_0, \epsilon)$. Now let $f_\lambda(z) = \lambda^{n_Z/2m} f(\lambda^{1/m} z)$, so that $\|f_\lambda\|_{L^2} = 1$ and $\text{supp}(f_\lambda) \subseteq B(\lambda^{-1/m} z_0, \lambda^{-1/m} \epsilon)$. Then

$$\begin{aligned} T_\lambda f_\lambda(x) &= \int e^{i\lambda S(x,z)} a(x,z) f_\lambda(z) dz \\ &= \int e^{iS(\lambda^{1/m} x, \lambda^{1/m} z)} a(x, \lambda^{-1/m} \lambda^{1/m} z) f(\lambda^{1/m} z) \lambda^{-n_Z/2m} \lambda^{n_Z/m} dz \\ &= \lambda^{-n_Z/2m} \int e^{iS(\lambda^{1/m} x, z')} a(x, \lambda^{-1/m} z') f(z') dz', \end{aligned}$$

so that $|T_\lambda f_\lambda(x)| \geq C\lambda^{-n_Z/2m}$ for $x \in B(\lambda^{-1/m} x_0, \lambda^{-1/m} \epsilon)$. Hence, $\|T_\lambda f_\lambda\| \geq C\lambda^{-n_Z/2m} (\lambda^{-n_X/m})^{1/2}$ and thus $\|T_\lambda\| \geq C\lambda^{-(n_X+n_Z)/2m}$.

For (4.2), note that if $\text{rank}(S''_{x,z}(x_0, z_0)) = n_Z$, then we can make a linear change of variables so that $x = (x', x'') \in \mathbb{R}^{n_X-n_Z} \times \mathbb{R}^{n_Z}$ and $\det S''_{x'',z}(x_0, z_0) \neq 0$. For each x' near x'_0 , the operator

$$f \rightarrow \left(T_\lambda^{x'} f \right) (x'') := \int e^{i\lambda S(x', x'', z)} a(x', x'', z) f(z) dz$$

is as in [7] and so $\|T_\lambda^{x'}\|_{L^2(\mathbb{R}^{n_Z}) \rightarrow L^2(\mathbb{R}^{n_Z})} \geq C\lambda^{-n_Z/2}$. Hence, $\|T_\lambda\|$ satisfies the same lower bound. \square

4.2. Optimality of assumptions. The focus of this work is establishing the decay estimates for oscillatory integral operators whose phase functions are generic homogeneous polynomials. However, determining exactly which homogeneous polynomial phases enjoy the same decay rates as those for generic phases seems to be

a difficult problem. For Thm. 1.4, we note in passing that for a direct sum of two generic cubics in $(1+1)$ -dimensions,

$$(4.3) \quad S(x, z) = x_1 z_1^2 + x_1^2 z_1 + x_2 z_2^2 + x_2^2 z_2 \quad ,$$

iterating the one-dimensional result [8],[9], shows that $\|T_\lambda\| \leq (C\lambda^{-1/3})^2 = C^2\lambda^{-2/3}$. This is the same rate as for phase functions covered by Thm. 1.4, and, although (1.9) is satisfied, the matrices in (1.10) are zero and $\Sigma \setminus (0,0)$ is not smooth, but rather a normal crossing. Thus, the hypotheses of Thm. 1.4 are not necessary for the $2/3$ decay rate to hold.

4.3. Newton distance and decay. We now make a few observations about the relationship between the decay rates in Theorems 1.1–1.4 and the Newton decay rate. If $S(x, z) \in C^\omega(\mathbb{R}^{n_X+n_Z})$ with Taylor series $\sum c_{\alpha,\beta} x^\alpha z^\beta$ having no pure x - or z -terms, let

$$\mathcal{N}_0(S) = \text{convex hull} \left(\bigcup_{c_{\alpha,\beta} \neq 0} (\alpha, \beta) + \mathbb{R}_+^{n_X+n_Z} \right).$$

Then the *Newton polytope* of $S(x, z)$ (at $(0,0)$) is

$$(4.4) \quad \mathcal{N}(S) := \partial(\mathcal{N}_0(S)),$$

and the *Newton distance* $\delta(S)$ of S is then

$$(4.5) \quad \delta(S) := \inf\{\delta > 0 : (\delta, \dots, \delta) \in \mathcal{N}(S)\}.$$

One easily sees that if $S(x, z)$ is a homogeneous polynomial of degree m , then $\delta(S) \geq m/(n_X + n_Z)$.

In $(1+1)$ -dimensions, the decay rate of T_λ is determined in terms of the Newton distance of the phase; the following result from [11] is a considerable extension of Thm. A:

Theorem C (Phong and Stein). *If $S \in C^\omega(\mathbb{R}^{1+1})$ with Newton distance $\delta = \delta(S)$, then $\|T_\lambda\| \leq C\lambda^{-\frac{1}{2\delta}}$.*

Referring to $1/(2\delta)$ as the *Newton decay rate* of $S(x, z)$, we now show that the decay rates in Thm. 1.1 (in the equidimensional case), Thm. 1.3 and Thm. 1.4 are equal to the Newton decay rate, when the decay rate is less than $n_Z/2$.

Proposition 4.2. *If $n_X = n_Z = n$ and $S(x, z)$ is nondegenerate as described in the hypothesis of Thm. 1.1, then $\delta(S) = m/2n$.*

Proof. Since $\det S''_{xz}(x, z) \neq 0$ for all $(x, z) \neq (0, 0)$, this holds in particular on all $2n$ of the coordinate axes away from $(0, 0)$. Consider the x_1 -axis, where $x_2 = \dots = x_n = z_1 = \dots = z_n = 0$. Let $A = (a_{ij}) = S''_{xz}(x_1, 0, \dots, 0)$. Since $\det A \neq 0$, for some permutation $\sigma \in S_n$, we have $a_{1\sigma(1)} \dots a_{n\sigma(n)} \neq 0$. Since

$$\begin{aligned} a_{ij} &= S''_{x_i x_j} |_{x_1\text{-axis}} \\ &= (\text{coefficient of } x_1^{m-2} x_i z_j \text{ in } S(x, z)) \times \begin{cases} m-1, & i=1 \\ 1, & i \neq 1, \end{cases} \end{aligned}$$

so the coefficient of $x_1^{m-2}x_i z_{\sigma(i)} \neq 0$, $1 \leq i \leq n$. This implies that for every $1 \leq i \leq n$,

$$\begin{bmatrix} e_i \\ \dots \\ e_{\sigma(i)} \end{bmatrix} + \begin{bmatrix} m-2 \\ 0 \\ \vdots \\ 0 \\ \dots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathcal{N}_0(S),$$

where $\{e_i\}$ is the standard basis for \mathbb{R}^n . Taking the $(\frac{1}{n}, \dots, \frac{1}{n})$ -weighted convex combination of these, we see that

$$\frac{1}{n} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \dots \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} m-2 \\ 0 \\ \vdots \\ 0 \\ \dots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathcal{N}_0(S).$$

Repeating this argument for the other $2n - 1$ coordinate axes and then taking the $(\frac{1}{2n}, \dots, \frac{1}{2n})$ -weighted convex combination, we find that

$$\frac{1}{n} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \frac{1}{2n} \begin{bmatrix} m-2 \\ \vdots \\ m-2 \end{bmatrix} = \frac{m}{2n} \begin{bmatrix} m \\ \vdots \\ m \end{bmatrix} \in \mathcal{N}_0(S).$$

Hence, $\delta(S) \leq m/2n$; but, as noted earlier, $\delta(S) \geq m/2n$, so that $\delta(S) = m/2n$. \square

Similarly, we next show that the decay rate in Thm. 1.3 equals the Newton decay rate for large m :

Proposition 4.3. *If $S(x, z) \in \mathfrak{S}^m \mathbb{R}^{n_X+n_Z}$ satisfies the rank one condition (1.5), and either $m \geq 5$, or $n_X = n_Z = 2$ and $m \geq 4$, then $\delta(S) = \frac{m}{n_X+n_Z}$.*

Proof. As in the proof of Prop. 4.2, we consider S''_{xz} evaluated along each of the $n_X + n_Z$ coordinate axes away from $(0, 0)$. For $1 \leq k \leq n_X$, on the x_k -axis the only terms in S''_{xz} which are $\neq 0$ are of the form $c_{ij}x_k^{m-2}$, and there must be at least one with $c_{ij} \neq 0$, since $\text{rank}(S''_{xz}) \geq 1$. Hence, $\mathcal{N}_0(S)$ contains vectors of the form

$$\vec{A}_k := \begin{bmatrix} (m-2)e_k \\ 0 \end{bmatrix} + \begin{bmatrix} e_{i_k} \\ e_{j_k} \end{bmatrix}, 1 \leq k \leq n_X,$$

with $i_k \leq n_X < j_k$, where $\{e_i\}_{i=1}^{n_X+n_Z}$ is the standard basis of column vectors. By considering S''_{xz} along the z_l -axis, $\mathcal{N}_0(S)$ also contains

$$\vec{A}_l := \begin{bmatrix} 0 \\ (m-2)e_l \end{bmatrix} + \begin{bmatrix} e_{i_l} \\ e_{j_l} \end{bmatrix}, n_X + 1 \leq l \leq n_X + n_Z,$$

with $i_l \leq n_X < j_l$. Forming the $(n_X + n_Z) \times (n_X + n_Z)$ matrix A with these columns, we have $A = (m - 2)I + R$, with each column of R having one 1 among the first n_X rows and one 1 among the last n_Z rows. We claim that if $m \geq 5$ then A is nonsingular. If not, consider a nontrivial linear combination, $\sum_{j=1}^{n_X+n_Z} c_j \vec{A}_j = \vec{0}$. Note that the sum of the elements in each column \vec{A}_j equals m ; hence, $\sum c_j = 0$. Suppose that there are k negative c_j 's and $n_X + n_Z - k$ nonnegative c_j 's; for notational convenience only, we may assume that $c_1, \dots, c_k < 0$ and then

$$\sum_{j=1}^k c_j = - \sum_{j=k+1}^{n_X+n_Z} c_j = -C$$

for some $C > 0$. Now consider the sum of all $k(n_X + n_Z)$ entries in the first k rows of $\sum_{j=1}^{n_X+n_Z} c_j \vec{A}_j$, which must equal 0. The contribution from the first k columns must be $\leq -(m - 2)C$, since each c_j multiplies the $m - 2$ in the j^{th} row, and there may be other positive multiples of $c_j < 0$ as well, coming from the 1's in the j^{th} column. On the other hand, the contribution from the $c_j \vec{A}_j$ with $k + 1 \leq j \leq n_X + n_Z$ is $\leq 2C$, since there are at most two 1's among the first k rows of the j^{th} column. Thus, $0 \leq 2C - (m - 2)C = (4 - m)C$, which is a contradiction if $m \geq 5$.

To prove Prop. 4.3, it suffices to show that

$$\vec{A}_0 := \frac{m}{n_X + n_Z} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

lies in the convex hull of the \vec{A}_j , since this implies that $\delta(S) \leq \frac{m}{n_X+n_Z}$ and \geq holds because of the homogeneity of $S(x, z)$. Since A is nonsingular, there exist unique $b_j \in \mathbb{R}$ such that $\vec{A}_0 = \sum b_j \vec{A}_j$. Using again the fact that the sum of the entries in each \vec{A}_j equals m , we see that $\sum b_j = 1$; hence, it merely remains to show that the b_j are nonnegative. If not, we reason as above: suppose that $b_j < 0, 1 \leq j \leq k$ and $b_j \geq 0, k + 1 \leq j \leq n_X + n_Z$; then

$$\sum_{j=1}^k b_j = 1 - \sum_{k+1}^{n_X+n_Z} b_j = 1 - B$$

for some $B > 1$. Again consider the sum of the terms in the first k rows of $\sum b_j \vec{A}_j$. The sum of the terms in the first k columns is $\leq (m - 2)(1 - B)$, while the sum of the remaining terms is either $\leq B$ (if $k = 1$) or $\leq 2B$ (if $k \geq 2$), since there are at most two 1's in each column of A . Hence, if $k = 1$,

$$1 \leq \frac{m}{n_X + n_Z} = \text{sum of entries in first row of } \sum b_j \vec{A}_j \\ \leq (m - 2)(1 - B) + B$$

which implies $0 \leq (3 - m)(B - 1)$, whence $m \leq 3$, a contradiction. Similarly, if $k \geq 2$,

$$k \leq \frac{km}{n_X + n_Z} \leq (m - 2)(1 - B) + 2B,$$

which implies $0 \leq k + 2 \leq (m - 4)(1 - B)$, whence $m \leq 4$, a contradiction. Hence, all of the b_j are nonnegative, proving that \vec{A}_0 is in the convex hull of the \vec{A}_j and thus $\delta(S) = \frac{m}{n_X+n_Z}$, finishing the proof for $m \geq 5$.

For $m = 4$, the proof that A is nonsingular breaks down if $k \geq 2$. If $n_X = n_Z = 2$, interchanging the analysis of positive and negative coefficients, we see that there must be two of each if A is to be singular, and then without loss of generality one can see that A has the form

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

Since $[1, 1, 1, 1]^t$ is the average of the columns, it follows that $\delta(S) \leq 1 = \frac{m}{n_X + n_Z}$. \square

Finally, we show that for cubics on \mathbb{R}^{2+2} such that (5.5) holds, the Newton decay rate is $2/3$:

Proposition 4.4. *If $S(x, z) \in \mathfrak{S}^3\mathbb{R}^{2+2}$ is such that $\tilde{\Sigma}$ is smooth, then $\delta(S) = 3/4$.*

Proof. The smoothness of Σ away from the origin implies that

$$(4.6) \quad \{d_{x,z}S''_{x_1z_1}, d_{x,z}S''_{x_1z_2}, d_{x,z}S''_{x_2z_1}, d_{x,z}S''_{x_2z_2}\}$$

is linearly independent. Thus, the four covectors in (4.6) have four distinct components corresponding to some permutation of $\{x_1, x_2, z_1, z_2\}$, which are $\neq 0$. Assume without loss of generality that $d_{x_1}S''_{x_1z_1} \neq 0$. Then

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{N}_0(S).$$

Continuing with the derivatives $d_{x_2}, d_{z_1}, d_{z_2}$ of some permutation of $\{S''_{x_1z_2}, S''_{x_2z_1}, S''_{x_2z_2}\}$ and taking the $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ -weighted convex combination, we see that

$$\frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \mathcal{N}_0(S).$$

Hence, $\delta \leq 3/4$, and again $\delta \geq 3/4$ by homogeneity. \square

In general however, the relationship between the decay rate and Newton distance in several variables is not clear. In the cases we considered above, the Newton distances are invariant under linear transformations in x and linear transformations in z , but in general this is not true. For example, if $S(x, z) = x_1^2z_1 + x_1z_1^2 \in \mathfrak{S}^3\mathbb{R}^{2+2}$, the Newton distance of $S(x, z)$ is $\frac{3}{2}$, which changes to $\frac{3}{4}$ if one rotates in x and z separately by angles $\theta_1, \theta_2 \notin \pi\mathbb{Z}$. Since the decay rate is invariant under linear transformations in x and linear transformations in z , the direct relationship between Newton distance and decay rate of oscillatory integral operators that holds in (1+1)-dimensions and in Theorems 1.1, 1.3 and 1.4, does not hold for general phases in higher dimensions. For $S(x, z) = x_1^2z_1 + x_1z_1^2$, the maximum of all the Newton distances of the phase function after composition with linear transformations in x and linear transformations in z is $\frac{3}{2}$, and this gives the correct decay rate. Thus we are led to the following definition and conjecture; these are related to a condition for scalar oscillatory integrals with real-analytic phases due to Varchenko [19].

Definition. Let $S(x, z) \in \mathfrak{S}^m \mathbb{R}^{n_x + n_z}$. The modified Newton distance of S is

$$(4.7) \quad \delta_{\text{mod}}(S) = \sup \{ \delta(S(Ax, Bz)) : A \in GL(n_x), B \in GL(n_z) \}.$$

Conjecture. If $S \in \mathfrak{S}^m \mathbb{R}^{n_x + n_z}$, then

$$\|T_\lambda\| \leq C \lambda^{-1/(2\delta_{\text{mod}}(S))} (\log(\lambda))^p$$

for some $p \geq 0$.

As further evidence for the conjecture, we consider phase functions in $(2+2)$ -dimensions associated with pencils of homogeneous forms. Let $S(x, z) = x_1 \phi_1(z) + x_2 \phi_2(z)$, where $\phi_1(z)$ and $\phi_2(z)$ are homogeneous polynomials on \mathbb{R}^2 of the same degree. Fu [1] obtained decay estimates for such phase functions when $\phi_1(z)$ and $\phi_2(z)$ satisfy some generic conditions. (See also [5] for some motivation coming from integral geometry for studying such families of phase functions). Since $\phi_1(z)$ and $\phi_2(z)$ are homogeneous polynomials on \mathbb{R}^2 , they can be factored into linear factors over \mathbb{C} . For $(a, b) \in \mathbb{R}^2 \setminus (0, 0)$, denote the minimum of the multiplicities of $az_1 + bz_2$ in ϕ_1 and ϕ_2 by $m(a, b)$. Let

$$(4.8) \quad s = \max_{(a,b) \in \mathbb{R}^2 \setminus (0,0)} m(a, b).$$

The following result supports the statement of the conjecture.

Proposition 4.5. Let $S(x, z) = x_1 \phi_1(z) + x_2 \phi_2(z)$, where ϕ_1 and ϕ_2 are homogeneous polynomials of degree d . Then for s as in (4.8),

- (a) $\|T_\lambda\| \leq C \lambda^{-r} (\log \lambda)$ with $r = \min(\frac{1}{d}, \frac{1}{2s})$. The bound is optimal except possibly the logarithmic term, in the sense that $\|T_\lambda\| \geq c \lambda^{-r}$.
- (b) The exponent r defined above equals $1/(2\delta_{\text{mod}}(S))$.

Remark : It should be pointed out that (up to the log term) the proposition above improves upon an earlier result of Fu [1, Thm. 1.2], where the decay exponent $-1/d$ (but without any logarithmic growth) was obtained only under generic conditions on ϕ_1 and ϕ_2 . Here we have placed no such restrictions on these functions. Furthermore, our proof can easily be adapted to show that the log term can be dispensed with under the generic conditions imposed in [1].

Proof. It is sufficient to prove that for each point in the unit circle of \mathbb{R}_Z^2 , an operator supported in any one of its (small enough) convex conic neighborhood has the desired decay rate. Since the decay rate does not change under linear transformations in z , we can transform the point to $(0, 1)$, and it suffices to prove it for $(0, 1)$.

Let $m_0 = m(0, 1)$. Then $m_0 \leq s$. Suppose that $\phi_1(z) = z_2^{m_0} \varphi_1(z)$ and $\phi_2(z) = z_2^{m_0} \varphi_2(z)$, so that at least one of φ_1 and φ_2 is not divisible by z_2 . Then the minimum of the multiplicities of z_2 in $\partial\phi_1/\partial z_2$ and $\partial\phi_2/\partial z_2$ is $m_0 - 1$.

We decompose the conic neighborhood of $(0, 1)$ into dyadic rectangles, where

$$(4.9) \quad |z_i| \sim 2^{-j_i}, \quad i = 1, 2, \quad j_2 - j_1 \gg C.$$

Then

$$T_\lambda = \sum_{j_1, j_2} T_\lambda^{j_1, j_2},$$

where $T_\lambda^{j_1, j_2}$ is an oscillatory integral operator with the same phase function as T_λ , but with amplitude supported in the dyadic rectangle (4.9). Further, the discussion in the preceding paragraph implies that

$$\left| \frac{\partial \phi_1}{\partial z_2} \right| + \left| \frac{\partial \phi_2}{\partial z_2} \right| \sim 2^{-(d-m_0)j_1} 2^{-(m_0-1)j_2}.$$

Without loss of generality assume that $\partial \phi_1 / \partial z_2$ satisfies the above estimate. Therefore using the operator Van der Corput lemma in the (x_1, z_2) variables, and Young's inequality in (x_2, z_1) , we obtain

$$(4.10) \quad \|T_\lambda^{j_1, j_2}\| \lesssim \left(\lambda^{-\frac{1}{2}} 2^{\frac{(d-m_0)j_1}{2}} 2^{\frac{(m_0-1)j_2}{2}} \right) 2^{-\frac{j_1}{2}} = \lambda^{-\frac{1}{2}} 2^{\frac{d-1-m_0}{2}j_1} 2^{\frac{(m_0-1)j_2}{2}}.$$

On the other hand, Young's inequality in all variables yields,

$$(4.11) \quad \|T_\lambda^{j_1, j_2}\| \lesssim 2^{-\frac{j_1+j_2}{2}}.$$

Summing (4.10) and (4.11) over $j_1 + j_2 = j$, we obtain

$$\sum_{j_1+j_2=j} \|T_\lambda^{j_1, j_2}\| \lesssim \left\{ \begin{array}{l} \left\{ \begin{array}{l} j \lambda^{-\frac{1}{2}} 2^{\frac{d-2}{4}j} \text{ if } m_0 \leq \frac{d}{2} \\ j \lambda^{-\frac{1}{2}} 2^{\frac{m_0-1}{2}j} \text{ if } m_0 > \frac{d}{2} \end{array} \right\} \text{ from (4.10),} \\ j 2^{-\frac{j}{2}} \text{ from (4.11).} \end{array} \right\}$$

It follows that

$$\|T_\lambda\| \lesssim \begin{cases} \lambda^{-\frac{1}{2m_0}} \log \lambda & \text{if } m_0 > \frac{d}{2} \\ \lambda^{-\frac{1}{d}} \log \lambda & \text{if } m_0 \leq \frac{d}{2}. \end{cases}$$

This proves the first half of (a).

Test functions can be used to prove the optimality. We can assume that the amplitude $a(x, z)$ is bounded below by a positive constant in a small neighborhood of the origin. Choose a function f_λ such that

$$f_\lambda(z) = \begin{cases} 1 & \text{if } \lambda^{\frac{1}{d}}|z| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f_\lambda\|^2 \sim \lambda^{-\frac{2}{d}}$, while for ϵ_0 sufficiently small

$$|T_\lambda f_\lambda(x)| \geq c \lambda^{-\frac{2}{d}} \quad \text{for } |x| < \epsilon_0.$$

Therefore, $\|T_\lambda\| \geq \|T_\lambda f_\lambda\| / \|f_\lambda\| \gtrsim \lambda^{-2/d} / \lambda^{-1/d} = \lambda^{-1/d}$, and we have proved the sharpness of the decay exponent when $s \leq d/2$.

When $s > d/2$, we may assume that $s = m_0 = m(0, 1)$ after a linear transformation in z . Thus, $S(x, z) = z_2^s(x_1 \varphi_1(z) + x_2 \varphi_2(z))$, where φ_1 and φ_2 are homogeneous polynomials of degree $d - s$ and at least one of them is not a multiple of z_2 . Since

$$\lim_{z_2 \rightarrow 0} |\varphi_1(1, z_2)| + \lim_{z_2 \rightarrow 0} |\varphi_2(1, z_2)| > 0,$$

we can choose constants a and b such that

$$\lim_{z_2 \rightarrow 0} a \varphi_1(1, z_2) + b \varphi_2(1, z_2) \neq 0.$$

Therefore by the continuity of the phase function we can find small fixed constants c and $\epsilon > 0$ such that

$$c < |x_1 \varphi_1(z) + x_2 \varphi_2(z)| < c^{-1} \quad \text{for } |x - (a, b)| < \epsilon, |z - (1, 0)| < \epsilon.$$

Choose a function g_λ as follows,

$$g_\lambda(z) = \begin{cases} 1 & \text{if } \lambda|z_2|^s \leq \pi c/100, |z_1 - 1| < \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|g_\lambda\|^2 \sim \lambda^{-\frac{1}{s}}$, while $|T_\lambda g_\lambda(x)| \gtrsim \lambda^{-\frac{1}{s}}$ for $|x - (a, b)| < \epsilon$. Therefore $\|T_\lambda\| \gtrsim \lambda^{-1/(2s)}$, and we have proved the sharpness of the decay rate when $s > d/2$.

It remains to verify that $r = 1/(2\delta_{\text{mod}}(S))$. Suppose first $s \leq d/2$. It follows from the definition of s that for some $i = 1, 2$, the multiplicity of z_i in ϕ_i is $\leq d/2$. Without loss of generality, let us assume $i = 1$. Then $\mathcal{N}_0(S)$ contains a point of the form $(1, 0, d_1, d - d_1)$ with $d_1 \leq d/2$. Similarly, the common multiplicity of z_2 in ϕ_1 and ϕ_2 is $\leq d/2$. Therefore, there exists a point in $\mathcal{N}_0(S)$ of the form $(\kappa_0, 1 - \kappa_0, d_2, d - d_2)$ with $d_2 \geq d/2$ and $\kappa_0 = 0$ or 1 . Let $0 \leq \theta \leq 1$ be such that $\theta d_1 + (1 - \theta)d_2 = d/2$. By convexity, $(\theta, 1 - \theta, d/2, d/2) \in \mathcal{N}_0(S)$ if $\kappa_0 = 0$; and $(1, 0, d/2, d/2) \in \mathcal{N}_0(S)$ if $\kappa_0 = 1$. Since $d \geq 2$, and for any point (x, z) in $\mathcal{N}_0(S)$, the positive orthant with corner at (x, z) is also in $\mathcal{N}_0(S)$, we conclude that $(d/2, d/2, d/2, d/2) \in \mathcal{N}_0(S)$. Therefore, $\delta(S) \leq d/2$. On the other hand, by the homogeneity of ϕ_1 and ϕ_2 , $\delta(S) \geq d/2$. Since this argument applies for S composed with any linear transformation of the form $(x, z) \mapsto (Ax, Bz)$, we obtain $\delta_{\text{mod}}(S) = d/2$.

Next suppose that $s > d/2$. Denoting the multiplicity of z_i in $\phi_j(z)$ by d_{ij} , we identify four points in $\mathcal{N}_0(S)$, namely $(1, 0, d_{ij}, d - d_{ij})$, $1 \leq i, j \leq 2$. By the definition of s , $\min(d_{i1}, d_{i2}) \leq s$ for $i = 1, 2$. Therefore there exist numbers $d_1, d_2 \leq s$ such that $(1, 0, d_1, d - d_1), (0, 1, d - d_2, d_2) \in \mathcal{N}_0(S)$. The same argument as above then shows that $(s, s, s, s) \in \mathcal{N}_0(S)$ and $\delta_{\text{mod}}(S) \leq s$. On the other hand, let $az_1 + bz_2$ be a factor with multiplicity at least s in both ϕ_1 and ϕ_2 . By a linear transformation $z \mapsto w = \eta(z)$ where $w_1 = az_1 + bz_2$, we can assume that w_1 has multiplicities at least s in ϕ_1 and ϕ_2 . Then all points in $\mathcal{N}_0(S \circ \eta^{-1})$ are of the form $(1, 0, d_1, d - d_1)$ or $(0, 1, d_1, d - d_1)$, where $d_1 \geq s$. Hence $\delta(S \circ \eta^{-1}) \geq s$, and we have proved that $\delta_{\text{mod}}(S) = s$. This finishes the proof of Prop. 4.5. \square

5. CUBICS IN 2 + 2 DIMENSIONS

In this section, we show that the hypotheses of Thm. 1.4 hold for generic cubic phase functions $S \in \mathfrak{S}^3\mathbb{R}^{2+2}$ and give geometric interpretations of these conditions. By Prop. 3.1, it suffices to show that the corresponding conditions hold for generic $H \in \mathbb{M}_\mathfrak{h}[S^1\mathbb{R}^{2+2}]$ (which we now denote by $\mathbb{M}_\mathfrak{h}$ for simplicity).

Note that $f : \mathbb{M}_\mathfrak{h} \rightarrow S^2\mathbb{R}^{2+2}$, $f(H)(x, z) = \Phi(x, z) := \det H(x, z)$, is a polynomial mapping, as are the functions $p, r : S^2\mathbb{R}^{2+2} \rightarrow \mathbb{R}$ defined by $p(\Phi) = \det P$ and $r(\Phi) = \det R$, where $\Phi \in S^2\mathbb{R}^{2+2}$ is written as in (1.8). Thus, if $p \circ f$ is not identically zero, i.e., if there exists an $H^{(1)} \in \mathbb{M}_\mathfrak{h}$ such that $p(f(H^{(1)})) \neq 0$, then $p(f(H)) \neq 0$ for all H in some nonempty Zariski open subset $\mathcal{V}_1 \subseteq \mathbb{M}_\mathfrak{h}$. Similarly, if there is an $H^{(2)}$ such that $r(f(H^{(2)})) \neq 0$, then $r(f(H)) \neq 0$ for all H in a nonempty Zariski open subset $\mathcal{V}_2 \subseteq \mathbb{M}_\mathfrak{h}$. Now, on $\mathcal{V}_1 \cap \mathcal{V}_2$, $(P(f(H)))^{-1}$ and $(R(f(H)))^{-1}$ are rational matrix-valued functions of H , and

$$(5.1) \quad \det(P - QR^{-1}Q^t) \text{ and } \det(R - Q^tP^{-1}Q)$$

are rational, scalar-valued functions of H . Again, if we can find $H^{(3)}, H^{(4)} \in \mathcal{V}_1 \cap \mathcal{V}_2$ such that the expressions in (5.1) are nonzero for $f(H^{(3)}), f(H^{(4)})$ respectively, then

they are nonzero for H lying in nonempty Zariski open sets $\mathcal{V}_3, \mathcal{V}_4$ respectively. The resultants in (1.11), when applied to $f(H)$, are rational functions of H and, if nonzero for some $H^{(5)}, H^{(6)}$ respectively, are nonzero for H lying in Zariski open sets $\mathcal{V}_5, \mathcal{V}_6$ respectively. Finally, if we can find $H^{(7)}, H^{(8)} \in \mathcal{V}_1 \cap \mathcal{V}_3$ such that the resultants in (1.12) are nonzero for $H^{(7)}, H^{(8)}$ respectively, then they are nonzero for all H lying in Zariski open sets $\mathcal{V}_7, \mathcal{V}_8$ respectively. Thus, if such H^j exist for $1 \leq j \leq 8$, then for H in the dense open subset $\cap_{j=1}^8 \mathcal{V}_j \subseteq \mathbb{M}_b$, the hypotheses of Thm. 1.4 hold, and by Prop. 3.1, Thm. 1.4 applies to phase functions in an open dense subset of $\mathfrak{S}^3\mathbb{R}^{2+2}$.

If we take

$$(5.2) \quad S^0(x, z) = x_1(z_1^2 + z_2^2) + x_2 z_1 z_2 + z_1(2x_1^2 - x_2^2) + z_2(x_1^2 + 3x_2^2),$$

then $H^{(0)} := S_{xz}^{0''}$ simultaneously satisfies the conditions for $H^{(j)}$, $1 \leq j \leq 8$, as above and thus S^0 both satisfies Thm. 1.4 and shows that the hypotheses of Thm. 1.4 are satisfied by generic $S(x, z) \in \mathfrak{S}^3\mathbb{R}^{2+2}$.

In fact,

$$H^0(x, z) = \begin{bmatrix} 4x_1 + 2z_1 & 2x_1 + 2z_2 \\ z_2 - 2x_2 & 6x_2 + z_1 \end{bmatrix}$$

from which one obtains that $\Phi^0(x, z) = \det H^0(x, z)$ is given by 1.8 with

$$P = \begin{bmatrix} 0 & 14 \\ 14 & 0 \end{bmatrix}, Q = \begin{bmatrix} 4 & -1 \\ 12 & 0 \end{bmatrix}, R = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

It is then readily seen that P, Q and R satisfy the conditions corresponding to membership in \mathcal{V}_j , $1 \leq j \leq 8$.

The hypotheses of Thm. 1.4 have the following geometric interpretations and implications which will be useful below. The *critical variety* of the phase function S is

$$\Sigma = \{(x, z) : \det S_{xz}''(x, z) = 0\},$$

which has as defining function the quadratic form $\Phi(x, z)$ given by (1.8), represented by $\begin{bmatrix} P & Q \\ Q^t & R \end{bmatrix}$. But, if P and R are nonsingular, we have

$$(5.3) \quad \left| \begin{bmatrix} P & Q \\ Q^t & R \end{bmatrix} \right| = |P| \cdot |R - Q^t P^{-1} Q| = |P - Q R^{-1} Q^t| \cdot |R|,$$

so (1.9) and (1.10) imply that Φ is nondegenerate and $\tilde{\Sigma} := \Sigma \setminus (0, 0)$ is smooth. Note that if Φ is sign-definite, then $\tilde{\Sigma} = \emptyset$ and Thm. 1.1 applies, yielding the estimate $\|T_\lambda\| \leq C\lambda^{-2/3}$. Thus, we assume henceforth that Φ is indefinite and $\tilde{\Sigma} \neq \emptyset$. We will also need, for $0 < |\epsilon| < c \ll 1$, the family of smooth quadrics

$$\Sigma^\epsilon = \{(x, z) : \Phi(x, z) = \epsilon\},$$

and set $\Sigma^0 = \tilde{\Sigma}$ for convenience. Note that

$$\{(x, z) : d_x \Phi(x, z) = 0\} = \{Px + Qz = 0\} = \{x = -P^{-1}Qz\}$$

is a codimension two plane, as is $\{(x, z) : d_z \Phi(x, z) = 0\} = \{Q^t x + Rz = 0\} = \{z = -R^{-1}Q^t x\}$; since $P - QR^{-1}Q^t$ is nonsingular, their intersection is $(0, 0)$. Furthermore, $\Phi|_{\{d_x \Phi=0\}}$ is nondegenerate since, on $\{d_x \Phi = 0\}$,

$$\Phi(x, z) = \Phi(-P^{-1}Qz, z) = \frac{1}{2}z^t(R - Q^t P^{-1}Q)z$$

and $R - Q^t P^{-1} Q$ is nonsingular by (1.10). Geometrically, this means that Σ^ϵ is transverse to $\{d_x \Phi = 0\}$, denoted $\Sigma^\epsilon \pitchfork \{d_x \Phi = 0\}$. Similarly, $\Sigma^\epsilon \pitchfork \{d_z \Phi = 0\}$ since $P - QR^{-1}Q^t$ is nonsingular. Hence, if we let

$$(5.4) \quad \mathcal{L}_R^\epsilon = \Sigma^\epsilon \cap \{d_x \Phi = 0\} \text{ and } \mathcal{L}_L^\epsilon = \Sigma^\epsilon \cap \{d_z \Phi = 0\},$$

then \mathcal{L}_R^0 and \mathcal{L}_L^0 are unions of lines and, for $\epsilon \neq 0$, \mathcal{L}_R^ϵ , \mathcal{L}_L^ϵ are smooth curves which are graphs over conic sections in \mathbb{R}_z^2 , \mathbb{R}_x^2 respectively. Since $\{d_x \Phi = 0\} \cap \{d_z \Phi = 0\} = (0, 0)$, we have $\mathcal{L}_R^\epsilon \cap \mathcal{L}_L^\epsilon = \emptyset$. We can summarize the discussion so far by:

Lemma 5.1. *Under assumptions (1.9) and (1.10),*

$$(5.5) \quad \Sigma^\epsilon \text{ is a smooth quadric in } \mathbb{R}^{2+2} \setminus (0, 0);$$

$$(5.6) \quad \mathcal{L}_R^\epsilon \text{ and } \mathcal{L}_L^\epsilon \text{ are unions of smooth curves ;}$$

$$(5.7) \quad \mathcal{L}_R^\epsilon \cap \mathcal{L}_L^\epsilon = \emptyset.$$

The significance of \mathcal{L}_R^ϵ and \mathcal{L}_L^ϵ is further explained by the following.

Lemma 5.2. *Let $\pi_R : \mathbb{R}^{2+2} \rightarrow \mathbb{R}_z^2$ and $\pi_L : \mathbb{R}^{2+2} \rightarrow \mathbb{R}_x^2$ denote the natural projections to the right and left. Then $\pi_R|_{\Sigma^\epsilon}, \pi_L|_{\Sigma^\epsilon} : \Sigma^\epsilon \rightarrow \mathbb{R}^2$ are submersions with folds, with critical sets \mathcal{L}_R^ϵ and \mathcal{L}_L^ϵ respectively.*

Proof. (For the definition and properties of a submersion with folds see for example [6, p. 87].) We only consider $\pi_R|_{\Sigma^\epsilon}$, since $\pi_L|_{\Sigma^\epsilon}$ is handled similarly. For $(x, z) \in \Sigma^\epsilon$,

$$T_{(x,z)}\Sigma^\epsilon = \{(\Delta x, \Delta z) : \langle d_x \Phi, \Delta x \rangle + \langle d_z \Phi, \Delta z \rangle = 0\},$$

so $\pi_R|_{\Sigma^\epsilon}$ is a submersion on $\Sigma^\epsilon \setminus \mathcal{L}_R^\epsilon = \{d_x \Phi(x, z) \neq 0\}$ by the implicit function theorem. At \mathcal{L}_R^ϵ ,

$$T_{(x,z)}\Sigma^\epsilon = T_x \mathbb{R}^2 \oplus (d_z \Phi)^\perp,$$

so $\dim \ker d\pi_R = \dim T_x \mathbb{R}^2 \oplus (0) = 2$. Hence, $d\pi_R$ drops rank by one at the codimension two submanifold \mathcal{L}_R^ϵ . Furthermore, since $\mathcal{L}_R^\epsilon = \{(x, z) \in \Sigma^\epsilon : \Phi'_{x_1} = \Phi'_{x_2} = 0\}$, we have

$$(\ker d\pi_R) \pitchfork \mathcal{L}_R^\epsilon \iff \begin{vmatrix} \Phi''_{x_1 x_1} & \Phi''_{x_2 x_1} \\ \Phi''_{x_1 x_2} & \Phi''_{x_2 x_2} \end{vmatrix} \neq 0.$$

But the righthand side is just $|P|$, which is nonzero by (1.9). Finally, we need to show that $d\pi_R$ drops rank *simply* at \mathcal{L}_R^ϵ ; this means that the ideal of smooth functions generated by the 2×2 minors of $d\pi_R$ is equal to the ideal of smooth functions vanishing on \mathcal{L}_R^ϵ . A frame for $T_{(x,z)}\Sigma^\epsilon$ consisting of essentially unit vectors is $\{V_0, V_1, V_2\}$, where

$$(5.8) \quad V_0 = \left((0, 0), \frac{(d_z \Phi)^\perp}{|d_z \Phi|} \right),$$

$$(5.9) \quad V_1 = \left((1, 0), \left(-\Phi'_{x_1} \frac{d_z \Phi}{|d_z \Phi|^2} \right) \right), \text{ and}$$

$$(5.10) \quad V_2 = \left((0, 1), \left(-\Phi'_{x_2} \frac{d_z \Phi}{|d_z \Phi|^2} \right) \right).$$

Since $d_z \Phi \neq 0$ near \mathcal{L}_R^ϵ , we have

$$d\pi_R(V_0 \wedge V_1) = \frac{(d_z \Phi)^\perp}{|d_z \Phi|} \wedge \left(-\Phi'_{x_1} \frac{d_z \Phi}{|d_z \Phi|^2} \right) \simeq \frac{\Phi'_{x_1}}{|d_z \Phi|} \left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \right)$$

and

$$d\pi_R(V_0 \wedge V_2) = \frac{(d_z \Phi)^\perp}{|d_z \Phi|} \wedge \left(-\Phi'_{x_2} \frac{d_z \Phi}{|d_z \Phi|^2} \right) \simeq \frac{\Phi'_{x_2}}{|d_z \Phi|} \left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \right),$$

where \simeq means that the two-vectors are smooth, nonvanishing multiples of each other. Thus, the ideal of 2×2 minors contains Φ'_{x_1} and Φ'_{x_2} ; since these generate the ideal of \mathcal{L}_R^ϵ , the two ideals are the same. \square

Locally, up to diffeomorphisms in the domain and range spaces, there exist two local normal forms [6, p. 88] for the submersion with folds $\pi_R : \Sigma^\epsilon \longrightarrow \mathbb{R}_z^2$, namely

$$\pi_R(t_1, t_2, t_3) = (t_1, t_2^2 \pm t_3^2)$$

with respect to suitable coordinates. If we restrict to $\frac{1}{2} \leq |(x, z)| \leq 2$ and $|\epsilon| \leq c$, then the changes of variables range over bounded sets in C^∞ . Thus, if $\mathbf{Q} \subset \mathbb{R}^{2+2}$ is a cube of side length ϱ , centered at $c(\mathbf{Q}) = (c_x(\mathbf{Q}), c_z(\mathbf{Q})) \in \Sigma^\epsilon$ and at distance δ from \mathcal{L}_R^ϵ , with $\varrho \leq c_0 \delta$, then $c_1 \mathbf{R}_{c_z(\mathbf{Q})} \subset \pi_R(\mathbf{Q}) \subset c_2 \mathbf{R}_{c_z(\mathbf{Q})}$, with $\mathbf{R}_{c_z(\mathbf{Q})} \subset \mathbb{R}_z^2$ a rectangle centered at $c_z(\mathbf{Q})$, of side lengths $\varrho \times \varrho^2$ if $c_3 \delta \leq \varrho \leq c_0 \delta$ and $\varrho \times (\delta \varrho)$ if $0 < \varrho \leq c_3 \delta$, and with major axis parallel to $(d_z \Phi)^\perp$ by (5.8). On the other hand, $\pi_L|_{\Sigma^\epsilon}$ is a submersion near \mathcal{L}_R^ϵ by (5.7), so $\pi_L(\mathbf{Q}) \subset \mathbb{R}_x^2$ is essentially a square of side length ϱ centered at $c_x(\mathbf{Q})$. Since $d\Phi$ is homogeneous of degree 1, we obtain:

Lemma 5.3. *Let $\mathbf{Q} \subset \mathbb{R}^{2+2}$ be a cube of side length ϱ centered at a point $c(\mathbf{Q}) \in \Sigma^\epsilon$ and with $0 < \varrho \leq c_0 \delta \leq c'_0 r$, where $\delta = \text{dist}(c(\mathbf{Q}), \mathcal{L}_R^\epsilon)$ and $r = |c(\mathbf{Q})|$. Then*

$$(5.11) \quad c_1 \mathbf{R}_{c_z(\mathbf{Q})} \subset \pi_R(\mathbf{Q}) \subset c_2 \mathbf{R}_{c_z(\mathbf{Q})}$$

where $\mathbf{R}_{c_z(\mathbf{Q})} \subset \mathbb{R}_z^2$ is a rectangle centered at $c_z(\mathbf{Q})$, of side lengths

$$\begin{cases} \varrho \times \varrho^2 & \text{if } c_3 \delta \leq \varrho \leq c_0 \delta \\ \varrho \times (\delta \varrho / r) & \text{if } 0 < \varrho \leq c_3 \delta, \end{cases}$$

and with major axis parallel to $(d_z \Phi)^\perp$. Also,

$$(5.12) \quad c_1 \mathbf{U}_{c_x(\mathbf{Q})} \subset \pi_L(\mathbf{Q}) \subset c_2 \mathbf{U}_{c_x(\mathbf{Q})}$$

where $\mathbf{U}_{c_x(\mathbf{Q})} \subset \mathbb{R}_x^2$ is a square centered at $c_x(\mathbf{Q})$ of side length ϱ .

We will also need to consider Σ^ϵ as an incidence relation between \mathbb{R}_z^2 and \mathbb{R}_x^2 . First, we define

$$(5.13) \quad \Gamma_R^\epsilon = \pi_R(\mathcal{L}_R^\epsilon) = \{z \in \mathbb{R}^2 : z^t(R - Q^t P^{-1} Q)z = \epsilon\}$$

and

$$(5.14) \quad \Gamma_L^\epsilon = \pi_L(\mathcal{L}_L^\epsilon) = \{x \in \mathbb{R}^2 : x^t(P - QR^{-1}Q^t)x = \epsilon\}.$$

Then $z \in \mathbb{R}^2 \setminus \Gamma_R^\epsilon \implies z$ is a regular value of $\pi_R|_{\Sigma^\epsilon}$, and $x \in \mathbb{R}^2 \setminus \Gamma_L^\epsilon \implies x$ is a regular value of $\pi_L|_{\Sigma^\epsilon}$. Thus, if we define

$$(5.15) \quad x\gamma^\epsilon = \{z \in \mathbb{R}^2 : (x, z) \in \Sigma^\epsilon\} = \{z : \Phi(x, z) = \epsilon\}, \text{ and}$$

$$(5.16) \quad \gamma_z^\epsilon = \{x \in \mathbb{R}^2 : (x, z) \in \Sigma^\epsilon\} = \{x : \Phi(x, z) = \epsilon\},$$

then ${}_x\gamma^\epsilon$ and γ_z^ϵ are smooth conic sections in \mathbb{R}^2 for all $x \in \mathbb{R}^2 \setminus \Gamma_L^\epsilon$, $z \in \mathbb{R}^2 \setminus \Gamma_R^\epsilon$ respectively. If $R - Q^t P^{-1} Q$ is sign-definite, then, depending on the sign of ϵ , Γ_R^ϵ is either empty or an ellipse with major- and minor-axes $\sim \epsilon^{1/2}$, and thus has curvature $\sim \epsilon^{-1/2}$. On the other hand, if $R - Q^t P^{-1} Q$ is indefinite, then Γ_R^ϵ is a hyperbola, with curvature $\sim \frac{\epsilon}{|z|^3}$. Similar comments hold for Γ_L^ϵ in terms of $P - QR^{-1}Q^t$.

6. DECOMPOSITION FOR CUBICS

6.1. Notation and preliminary reductions. We now turn to the decomposition that lies at the heart of the proof of Thm. 1.4. Since Φ vanishes to first order on $\tilde{\Sigma}$, S''_{xz} drops rank (by one) simply at $\tilde{\Sigma}$. Let $0 \leq \sigma_1(x, z) \leq \sigma_2(x, z)$ be the singular values of $S''_{xz}(x, z)$, i.e., the eigenvalues of $((S''_{xz})^t S''_{xz})^{1/2}$. The following conclusions are clear.

- (a) As functions of (x, z) , $\sigma_1(\cdot, \cdot)$ and $\sigma_2(\cdot, \cdot)$ are positively homogeneous of degree 1.
- (b) $\sigma_2(\cdot, \cdot)$ is smooth and $\sigma_2(x, z) \geq c|(x, z)|$.
- (c) $c_1|\Phi(x, z)| \leq \sigma_1(x, z)|(x, z)| \leq c_2|\Phi(x, z)|$. Thus σ_1 is essentially a (Lipschitz) defining function for Σ , i.e., $\sigma_1(x, z) \sim \text{dist}((x, z), \Sigma)$.

The proof of Thm. 1.4 involves several decompositions of the operator T . The successive decompositions are in terms of three indices k, j and ℓ , measuring the distance to $(0, 0)$, $\tilde{\Sigma}$ and \mathcal{L}_R^ϵ or \mathcal{L}_L^ϵ (for appropriate ϵ), respectively; each resulting piece is then decomposed further into cubes. To make this precise, let us first localize T to a neighborhood of Σ and away from the origin, where

$$1 \leq 2^{k+1}|(x, z)| \leq 2 \quad \text{and} \quad 1 \leq 2^{j+k+1}\sigma_1(x, z) \leq 2.$$

Then $T = \sum_{j, k \geq 0} T_{jk}$, where T_{jk} is of the same form (1.1) as T , but with amplitude

$$a_{jk}(x, z) = a(x, z)\psi(2^k|(x, z)|)\psi(2^{j+k}|\sigma_1(x, z)|).$$

Here $\psi(t) = \eta(t) - \eta(2t)$, and $\eta \in C_0^\infty(\mathbb{R})$ satisfies the properties: $\text{supp}(\eta) \subseteq [-2, 2]$, $\eta \equiv 1$ on $[-1, 1]$, so that $\sum_{k \in \mathbb{Z}} \psi(2^k \cdot) \equiv 1$ on $\mathbb{R} \setminus \{0\}$. Let us denote the support of a_{jk} by $\mathcal{O}(j, k)$, and set

$$(6.1) \quad \sigma_1 = c2^{-j-k}, \quad \sigma_2 = c2^{-k} \quad \text{and} \quad \epsilon = \sigma_1\sigma_2 = c^2 2^{-j-2k},$$

for some small constant $c > 0$ (depending only on the phase function S) to be chosen in the sequel. Thus, $\sigma_i(x, z) \sim \sigma_i$ for $(x, z) \in \mathcal{O}(j, k)$, $i = 1, 2$. Note that because of the small support of a and the remark following the proof of Thm. 1.3, it suffices to restrict attention only to non-negative indices k and j . Also, by remark (c) at the beginning of this section,

$$(6.2) \quad |\Phi| \sim \epsilon \quad \text{on } \mathcal{O}(j, k).$$

At the next step of the decomposition, the sets $\mathcal{O}(j, k)$, which are ‘‘hollow shells’’ of thickness σ_1 surrounding Σ , are divided into ‘‘curved slabs’’, with the dimensions of the slabs depending on their proximity to \mathcal{L}_R^ϵ and \mathcal{L}_L^ϵ . This is described below in greater detail. We begin with a few easy lemmas.

Lemma 6.1. *There exists a constant $C > 1$ such that if $(x, z) \in \mathcal{L}_R^\epsilon \cap \mathcal{O}(j, k)$, then $C^{-1}\sigma_2 \leq |z| \leq C\sigma_2$. Similarly, if $(x, z) \in \mathcal{L}_L^\epsilon \cap \mathcal{O}(j, k)$, then $C^{-1}\sigma_2 \leq |x| \leq C\sigma_2$.*

Proof. Recall the definition of \mathcal{L}_R^ϵ from (5.4). Since $2^{-k-1} \leq |(-P^{-1}Qz, z)| \leq C|z|$ on $\mathcal{L}_R^\epsilon \cap \mathcal{O}(j, k)$, the conclusion follows. \square

Lemma 6.2. *Suppose that $R - Q^t P^{-1}Q$ is sign-definite. Then $\mathcal{L}_R^\epsilon \cap \mathcal{O}(j, k) = \emptyset$. Similarly, $\mathcal{L}_L^\epsilon \cap \mathcal{O}(j, k) = \emptyset$ if $P - QR^{-1}Q^t$ is sign-definite.*

Proof. If $R - Q^t P^{-1}Q$ is sign-definite, there exists a constant $c_0 > 0$ such that $z^t(R - Q^t P^{-1}Q)z \geq c_0|z|^2$. Therefore, by Lemma 6.1,

$$\Phi(x, z) \Big|_{\{d_x \Phi = 0\}} = |z^t(R - Q^t P^{-1}Q)z| \geq c_0 c 2^{-2k} \gg \epsilon,$$

which contradicts (6.2). \square

Let us assume then that $R - Q^t P^{-1}Q$ and $P - QR^{-1}Q^t$ are sign-indefinite, so that $\mathcal{L}_R^\epsilon \cap \mathcal{O}(j, k)$ and $\mathcal{L}_L^\epsilon \cap \mathcal{O}(j, k)$ are nonempty. By Lemma 5.1, the curves given by $\mathcal{L}_R^\epsilon \cap \mathcal{O}(j, k)$ and $\mathcal{L}_L^\epsilon \cap \mathcal{O}(j, k)$ are disjoint. Let $z_0(1)$ and $z_0(2)$ be the two real and distinct nonzero solutions of $z^t(R - Q^t P^{-1}Q)z = 0$, $|z|^2 = 1$. Then $\Gamma_R^\epsilon = \pi_R \mathcal{L}_R^\epsilon$ is a hyperbola whose asymptotes point in the directions $z_0(1)$ and $z_0(2)$. Further, since $\epsilon \ll 2^{-2k}$, $\pi_R(\mathcal{L}_R^\epsilon \cap \mathcal{O}(j, k))$ consists of four disjoint curves, one from each branch of the two hyperbolas. Each curve is therefore almost parallel to either $\pm z_0(1)$ or $\pm z_0(2)$. An analogous statement applies to $\pi_L(\mathcal{L}_L^\epsilon \cap \mathcal{O}(j, k))$. One can therefore find a partition of unity in \mathbb{R}^4 , homogeneous of degree zero and subordinate to a finite family of overlapping cones $\{\mathcal{C}_i; 1 \leq i \leq N\}$, $N \leq 16$, such that each cone contains at most one connected component of $\mathcal{L}_R^\epsilon \cap \mathcal{O}(j, k)$ or $\mathcal{L}_L^\epsilon \cap \mathcal{O}(j, k)$. Using this partition of unity, T_{jk} splits into a finite number of summands, where the amplitude of the operator in the i th summand is supported in \mathcal{C}_i . Since interchanging the roles of x and z does not change the form of the operator T , it suffices to only deal with the situation where \mathcal{C}_i contains a branch of \mathcal{L}_R^ϵ . In what follows, the index i is fixed. So for simplicity, and by a slight abuse of notation, we drop this index and write the operator and its amplitude as T_{jk} and a_{jk} respectively.

The ‘‘curved slab’’ decomposition of T_{jk} is the following: we write

$$(6.3) \quad T_{jk} = \sum_{\ell=0}^j T_{\ell jk},$$

where $T_{\ell jk}$ is of the same form as T_{jk} but with amplitude

$$\begin{aligned} a_{\ell jk}(x, z) &= a_{jk}(x, z) \psi(2^{j-\ell+k} d(x, z)), \quad 0 < \ell \leq j, \\ a_{0jk}(x, z) &= a_{jk}(x, z) \eta(2^{j+k} d(x, z)). \end{aligned}$$

Here $d(x, z)$ denotes the distance of (x, z) from \mathcal{L}_R^ϵ . Fixing k and j , let $\mathcal{O}_\ell = \mathcal{O}_\ell(j, k)$ denote the support of $a_{\ell jk}$, and set

$$(6.4) \quad \sigma_0 = 2^{\ell-2j-k}.$$

The following lemma quantifies the ‘‘distortion’’ in the projections of \mathcal{O}_ℓ under π_R and π_L , and follows from the properties of submersion with folds.

Lemma 6.3. *There exists a constant $C > 0$ such that the π_R and π_L projections of \mathcal{O}_ℓ satisfy the containments below :*

$$\begin{aligned} \pi_R(\mathcal{O}_\ell) &\subseteq \{z \in \mathbb{R}_z^2 \mid C^{-1}\sigma_2 \leq |z| \leq C\sigma_2, \text{dist}(z, \Gamma_R^\epsilon) \leq C2^\ell \sigma_0\}, \\ \pi_L(\mathcal{O}_\ell) &\subseteq \{x \in \mathbb{R}_x^2 \mid |x| \leq C\sigma_2, \text{dist}(x, \pi_L(\mathcal{L}_R^\epsilon)) \leq C2^{\ell-j-k}\}. \end{aligned}$$

Proof. For the second containment, simply note that $|x| \leq c|z| \leq c\sigma_2$, and that projections decrease distances. For the first, use Lemma 6.1. Also note that since $d(x, z) \sim 2^{\ell-j-k}$ on \mathcal{O}_ℓ , the proof of Lemma 5.2 implies that $d\pi_R|_{\Sigma^\epsilon}$ acts as a projection from $\mathbb{R} \cdot V_0$ onto $\mathbb{R} \cdot (d_z\Phi)^\perp$ and as $\sim 2^{\ell-j}$ times the projection from $\text{span}(V_1, V_2)$ to $\mathbb{R} \cdot d_z\Phi$. \square

The decomposition in (6.3) is of course only meaningful if $R - Q^t P^{-1}Q$ is sign-indefinite. If it is sign-definite, then $d(x, z) \sim 2^{-k}$ on $\mathcal{O}(j, k)$, and the decomposition in ℓ is no longer necessary. All our subsequent analysis goes through in this case simply by setting $\ell = j$. In the sequel, we will only work with sign-indefinite $R - Q^t P^{-1}Q$, and leave the verification of the other (simpler) case to the reader.

The next section is devoted to the estimation of $\|T_{\ell j k}\|$. Although the symbols $a_{\ell j k}$ have slightly different forms for $\ell > 0$ and $\ell = 0$, they are treated similarly, and henceforth we give the argument only for $\ell > 0$, the proof for $\ell = 0$ going through with mainly notational changes.

Finally, we recall some standard terminology that will be used in the proof.

- Given a parallelepiped \mathcal{R} , its *dilate* $c\mathcal{R}$ is the parallelepiped with the same center as \mathcal{R} and each side scaled by a factor of c .
- A collection of sets $\tilde{\mathcal{Q}} = \{\tilde{Q}_i \mid i \in \mathcal{I}\}$ is said to be *essentially disjoint* if there exists a constant C (depending only on S) such that

$$\sup_{i \in \mathcal{I}} |\{i' \in \mathcal{I} \mid \tilde{Q}_i \cap \tilde{Q}_{i'} \neq \emptyset\}| \leq C.$$

6.2. Finer Decomposition of $T_{\ell j k}$ and Statement of the Main Result. The building blocks in the analysis of $T_{\ell j k}$ are cubes of sidelength approximately σ_1 . To make this precise, let us fix a set of σ_1 -separated points

$$(6.5) \quad \mathfrak{B}(\cdot) := \{(x_\beta, z_\beta) : \beta \in \mathfrak{b}\} \subseteq \mathcal{O}_\ell,$$

and define a family of cubes \mathcal{Q} as follows. A cube $\mathbf{Q} \in \mathcal{Q}$ if its sidelength is $C\sigma_1$ for some large constant C , and its center $c(\mathbf{Q}) = (c_x(\mathbf{Q}), c_z(\mathbf{Q})) = (x_\beta, z_\beta)$ for some $\beta \in \mathfrak{b}$. Clearly, \mathcal{Q} is essentially disjoint, and $\mathcal{O}_\ell \subseteq \bigcup_{\mathbf{Q} \in \mathcal{Q}} \mathbf{Q}$. We will see in §§6.3 that $|\mathcal{Q}| \sim 2^{2\ell+j}$. We will also describe in the same subsection a decomposition of \mathcal{Q} into a finite number of subcollections \mathcal{Q}_i ($1 \leq i \leq N$, for some $N \leq 16$) satisfying certain geometric properties.

Introducing a partition of unity subordinate to \mathcal{Q} , we can now write

$$T_{\ell j k} = \sum_{i=1}^N T_{\ell j k}^{(i)}, \quad \text{with} \quad T_{\ell j k}^{(i)} = \sum_{\mathbf{Q} \in \mathcal{Q}_i} \mathcal{T}_{\mathbf{Q}},$$

where the amplitudes $\{b_{\mathbf{Q}}\}$ of $\mathcal{T}_{\mathbf{Q}}$ satisfy

$$\text{supp}(b_{\mathbf{Q}}) \subseteq \mathbf{Q}, \quad \sum_{\mathbf{Q} \in \mathcal{Q}} b_{\mathbf{Q}} \equiv a_{\ell j k},$$

and the differentiability estimates

$$(6.6) \quad |\partial_{x,z}^\alpha b_{\mathbf{Q}}(x, z)| \leq C_\alpha 2^{(j+k)|\alpha|}, \quad |\alpha| \geq 0,$$

for some C_α independent of \mathbf{Q} . Using a version of the almost orthogonality lemma of Cotlar-Knapp-Stein[15, p. 318] we can estimate $\|T_{\ell j k}^{(i)}\|$ as follows :

$$\|T_{\ell j k}^{(i)}\| \leq \sup_{\mathbf{Q} \in \mathcal{Q}_i} \sum_{\mathbf{Q}' \in \mathcal{Q}_i} \|\mathcal{T}_{\mathbf{Q}} \mathcal{T}_{\mathbf{Q}'}^*\|^{\frac{1}{2}} + \sup_{\mathbf{Q} \in \mathcal{Q}_i} \sum_{\mathbf{Q}' \in \mathcal{Q}_i} \|\mathcal{T}_{\mathbf{Q}'}^* \mathcal{T}_{\mathbf{Q}}\|^{\frac{1}{2}}.$$

Thm. 1.4 is then a consequence of the following:

Proposition 6.4. *For \mathcal{Q}_i as above,*

$$(6.7) \quad \sum_{k,j,\ell} \sup_{\mathbf{Q} \in \mathcal{Q}_i} \sum_{\mathbf{Q}' \in \mathcal{Q}_i} \|\mathcal{T}_{\mathbf{Q}} \mathcal{T}_{\mathbf{Q}'}^*\|^{\frac{1}{2}} \leq C\lambda^{-\frac{2}{3}},$$

$$(6.8) \quad \sum_{k,j,\ell} \sup_{\mathbf{Q} \in \mathcal{Q}_i} \sum_{\mathbf{Q}' \in \mathcal{Q}_i} \|\mathcal{T}_{\mathbf{Q}'}^* \mathcal{T}_{\mathbf{Q}}\|^{\frac{1}{2}} \leq C\lambda^{-\frac{2}{3}}.$$

The proposition is proved in two parts. We prove (6.7) in §§7.3 and (6.8) in §§7.2.

6.3. Projections of \mathcal{Q} . To prepare for the proof of Prop. 6.4, we need an efficient way of indexing the cubes in \mathcal{Q} , and in particular of identifying when the x and z -supports of $b_{\mathbf{Q}}$ and $b_{\mathbf{Q}'}$ are disjoint. This leads us to investigate how the cubes in \mathcal{Q} project into \mathbb{R}_x^2 and \mathbb{R}_z^2 . Recalling the definition of the parameters $\sigma_0, \sigma_1, \sigma_2$ and ϵ from (6.1) and (6.4), the relevant facts are summarized in the lemmas below.

Lemma 6.5. *There exist constants $0 < c_i < 1 < C_i, i = 1, 2$ (depending only on the phase function S) with the following properties. Suppose that $\mathbf{Q} \in \mathcal{Q}$, with center $c(\mathbf{Q}) = (c_x(\mathbf{Q}), c_z(\mathbf{Q}))$.*

- (a) *Let \mathbf{R} be the rectangle (in \mathbb{R}_z^2) centered at $c_z(\mathbf{Q})$ with lengths σ_1 and σ_0 along the directions $c_z(\mathbf{Q})$ and $c_z(\mathbf{Q})^\perp$ respectively. Then $c_1\mathbf{R} \subseteq \pi_R\mathbf{Q} \subseteq C_1\mathbf{R}$.*
- (b) *Let \mathbf{U} be a square in \mathbb{R}_x^2 centered at $c_x(\mathbf{Q})$ with sidelength σ_1 . Then $c_2\mathbf{U} \subseteq \pi_L\mathbf{Q} \subseteq C_2\mathbf{U}$.*

Proof. For the proof of Lemma 6.5, we use Lemma 5.3 with $\varrho = \sigma_1, \delta = 2^{\ell-j-k}$ and $r = \sigma_2$, also noting that $\frac{z}{|z|} \cdot \frac{d_z\Phi}{|d_z\Phi|}$, which equals 0 on \mathcal{L}_R^0 by Euler's identity, is $O(2^{\ell-j})$ on \mathcal{O}_ℓ , so that z and $(d_z\Phi)^\perp$ are essentially parallel. \square

Lemma 6.6. *There exist constants C_3, C'_3 and C_4, C'_4 (depending only on S) with the following properties. Let \mathbf{R} be a rectangle in \mathbb{R}_z^2 centered at $z(\mathbf{R})$ whose dimensions along $z(\mathbf{R})$ and $z(\mathbf{R})^\perp$ are σ_1 and σ_0 respectively. Then,*

- (a) *The curve $\pi_R^{-1}(z(\mathbf{R})) \cap \Sigma^\epsilon \cap \mathcal{O}_\ell$ is of length $\leq C_3 2^{\ell-j-k}$.*
- (b) *The curve $\pi_L(\pi_R^{-1}(z(\mathbf{R})) \cap \mathcal{O}_\ell) = \pi_L(\mathcal{O}_\ell) \cap \gamma_{z(\mathbf{R})}^\epsilon$ is of length $\leq C'_3 2^{\ell-j-k}$.*
- (c) *The set $\pi_R^{-1}(\mathbf{R}) \cap \mathcal{O}_\ell$ is contained in a tubular neighborhood of the curve in (a), with the thickness of the tube comparable to σ_1 , i.e.,*

$$\sup \{ \text{dist}((x, z), \pi_R^{-1}(z(\mathbf{R})) \cap \mathcal{O}_\ell) : (x, z) \in \pi_R^{-1}(\mathbf{R}) \cap \mathcal{O}_\ell \} \leq C_4\sigma_1.$$

- (d) *The set $\pi_L(\pi_R^{-1}(\mathbf{R}) \cap \mathcal{O}_\ell)$ is contained in a tubular neighborhood of the curve in (b), with thickness of the tube comparable to σ_1 , i.e.,*

$$\sup \{ \text{dist}(x, \gamma_{z(\mathbf{R})}^\epsilon \cap \pi_L(\mathcal{O}_\ell)) : x \in \pi_L(\pi_R^{-1}(\mathbf{R}) \cap \mathcal{O}_\ell) \} \leq C'_4\sigma_1.$$

- (e) *The collection $\{\pi_L\mathbf{Q} \mid \mathbf{Q} \in \mathcal{Q}, c(\mathbf{Q}) \in \pi_R^{-1}(\mathbf{R}) \cap \mathcal{O}_\ell\}$ is essentially disjoint.*

Proof. The proofs of (a) and (b) are similar, so we concentrate on the latter. The curve γ_z^ϵ can be written as

$$\frac{1}{2}(x + P^{-1}Qz)^t P(x + P^{-1}Qz) = \epsilon - \frac{1}{2}z^t(R - Q^t P^{-1}Q)z.$$

In view of Lemma 6.3, (b) will be proved if we can show that the directions of the asymptotes of γ_z^ϵ , (namely p satisfying $p^t P p = 0$) are not the same as those of $\pi_L(\mathcal{L}_R^\epsilon)$ (namely $-P^{-1}Qz_0$, with z_0 satisfying $z_0^t(R - Q^t P^{-1}Q)z_0 = 0$). If indeed $p = -P^{-1}Qz_0$, then z_0 would also satisfy $z_0^t Q^t P^{-1}Q z_0 = 0$, and hence $z_0^t R z_0 = 0$. This would contradict the second nonvanishing resultant condition of (1.12). For part (c), we use the fact that off of $\mathcal{L}_R^\epsilon \cup \mathcal{L}_L^\epsilon$, z and $(d_z \Phi)^\perp$ are essentially parallel, and invoke the properties of $d\pi_R$ as outlined in the proof of Lemma 5.2. Part (d) follows since π_L decreases lengths. For part (e), we use the fact that $\ker(d\pi_L)$ and $\ker(d\pi_R)$ are one-dimensional subspaces spanned by linearly independent vectors. Thus, if \mathbf{Q} and \mathbf{Q}' are such that $c(\mathbf{Q}), c(\mathbf{Q}') \in \pi_R^{-1}\mathbf{R}$, then $c(\mathbf{Q}) - c(\mathbf{Q}')$ is essentially parallel to $\ker(d\pi_R)$, hence transverse to $\ker(d\pi_L)$, which implies that $\pi_L \mathbf{Q}$ and $\pi_L \mathbf{Q}'$ are essentially disjoint. \square

Lemma 6.7. *There exist constants $0 < c_3 < 1 < C_5, C_6$ depending only on S with the following properties. Let \mathbf{U} be a square in \mathbb{R}_x^2 centered at $x(\mathbf{U})$ with sidelength σ_1 . Then*

- (a) *The curve $\pi_L^{-1}(x(\mathbf{U})) \cap \Sigma^\epsilon \cap \mathcal{O}_\ell$ is of length $\leq C_5 2^{\ell-j-k}$.*
- (b) *The curve $\pi_R(\pi_L^{-1}(x(\mathbf{U})) \cap \mathcal{O}_\ell) = \pi_R(\mathcal{O}_\ell) \cap_{x(\mathbf{U})} \gamma^\epsilon$ is of length $\leq C_6 2^\ell \sigma_0$.*
- (c) *The curvature of the curve in (b) is bounded below by $c_3 \sigma_2^{-1}$.*
- (d) *The set $\pi_L^{-1}(\mathbf{U}) \cap \mathcal{O}_\ell$ is contained in a tubular neighborhood of the curve in (a) of thickness comparable to σ_1 .*
- (e) *The set $\pi_R(\pi_L^{-1}(\mathbf{U}) \cap \mathcal{O}_\ell)$ is contained in a tubular neighborhood of the curve in (b) of thickness comparable to σ_1 .*
- (f) *The collection $\{\pi_R \mathbf{Q} \mid \mathbf{Q} \in \mathcal{Q}, c(\mathbf{Q}) \in \pi_L^{-1}(\mathbf{U}) \cap \mathcal{O}_\ell\}$ is essentially disjoint.*

Proof. We only give the proof for parts (b) and (c), the proofs of the others being similar to their analogues in Lemma 6.6. For fixed x , the equation for ${}_x \gamma^\epsilon$ may be written as follows,

$$\frac{1}{2}(z + R^{-1}Q^t x)^t R(z + R^{-1}Q^t x) = \epsilon - \frac{1}{2}x^t(P - QR^{-1}Q^t)x.$$

Using Lemma 6.3, (b) follows from the second condition in (1.12), namely that the null directions of R and $R - Q^t P^{-1}Q$ are not the same. For (c), we use the second condition in (1.11) to conclude that $-P^{-1}Qz_0$ is not a null direction of $P - QR^{-1}Q^t$; therefore for $x \in \pi_L(\mathcal{O}_\ell)$,

$$|x^t(P - QR^{-1}Q^t)x| \sim 2^{-2k}, \quad \text{which implies} \quad |\epsilon - x^t(P - QR^{-1}Q^t)x| \sim \sigma_2^2.$$

The curvature of the hyperbola is therefore $\sim |\epsilon - x^t(P - QR^{-1}Q^t)x| / |z + R^{-1}Q^t x|^3 \gtrsim \sigma_2^2 \sigma_2^{-3} = \sigma_2^{-1}$, where at the last step we have used Lemma 6.3 to estimate the denominator. \square

Lemmas 6.5, 6.6 and 6.7 suggest two different schemes for enumerating the elements in \mathcal{Q} . For instance, we can first decompose \mathbb{R}_z^2 into $\sigma_1 \times \sigma_0$ rectangles of the form stated in part (a) of Lemma 6.5, and then count the cubes in the π_R -fiber of each such rectangle. Alternatively, we can start with a decomposition of \mathbb{R}_x^2 by a

family of σ_1 -squares, and count the cubes in the π_L -fiber of each square. We make this more precise below.

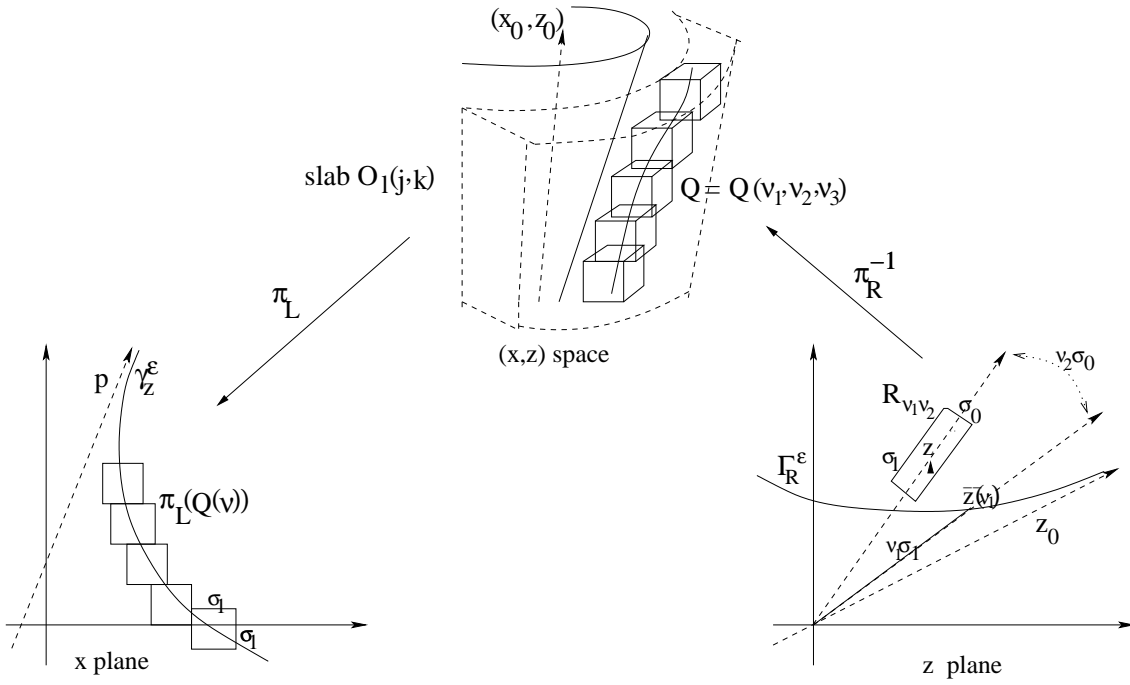
In the first scheme, $\pi_R(\mathcal{O}_\ell)$ is decomposed as follows. We pick σ_1 -separated points $\{\bar{z}(\nu_1)\}$ on $\pi_R(\mathcal{L}_R^\epsilon \cap \mathcal{O}_\ell)$, such that $|\bar{z}(\nu_1)| = \nu_1\sigma_1$, $C^{-1}2^j \leq \nu_1 \leq C2^j$. For ν_1 fixed, we choose σ_0 -separated points $\{z(\nu_1, \nu_2)\}$ on the circle centered at the origin of radius $\nu_1\sigma_1$, such that the angle between $\bar{z}(\nu_1)$ and $z(\nu_1, \nu_2)$ is $\nu_2\sigma_0$, $0 \leq \nu_2 \leq 2^\ell$. Then there exists a family of open rectangles $\{\mathbf{R}_{\nu_1, \nu_2}\}$ with the following properties : for each (ν_1, ν_2) , the rectangle $\mathbf{R}_{\nu_1, \nu_2}$ is centered at $z(\nu_1, \nu_2)$ and its dimensions along $z(\nu_1, \nu_2)$ and $z(\nu_1, \nu_2)^\perp$ are σ_1 and σ_0 respectively. The collection $\{\mathbf{R}_{\nu_1, \nu_2}\}$ is therefore essentially disjoint, and there exists a constant $C > 0$ such that $\pi_R\mathcal{O}_\ell = \bigcup_{\nu_1, \nu_2} C\mathbf{R}_{\nu_1, \nu_2}$. Let ν_3 index the cubes \mathbf{Q} whose centers lie in $\pi_R^{-1}(C\mathbf{R}_{\nu_1, \nu_2}) \cap \mathcal{O}_\ell$. For fixed (ν_1, ν_2) , the number of indices ν_3 is $\leq C2^\ell$, by Lemma 6.6.

It is clear that the enumeration scheme above assigns each cube in \mathbf{Q} a 3-tuple of indices $\nu = (\nu_1, \nu_2, \nu_3)$. However, a cube may have received multiple ν -s in this process. The number of such ν -s associated to a single cube is always bounded above by a fixed constant C . Selecting one representative ν from each such finite collection, we can ensure that every \mathbf{Q} has a unique index.

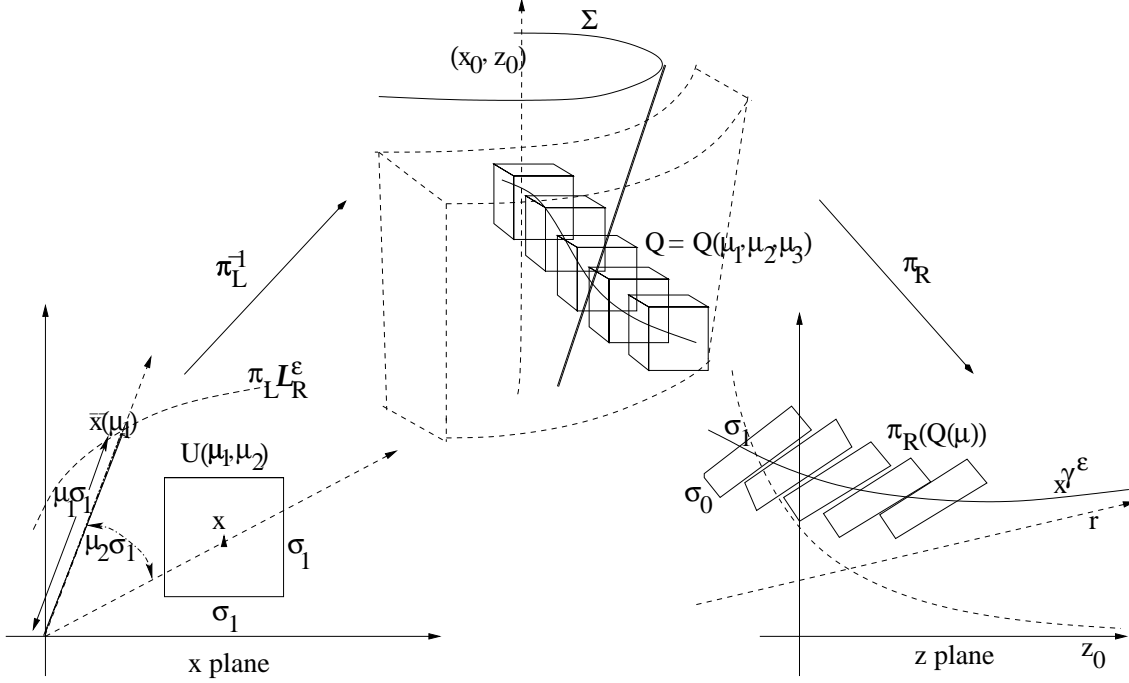
The second scheme for enumerating the elements of \mathcal{Q} is similar. Let $\{\bar{x}(\mu_1)\}$ be a collection of σ_1 -separated points on $\pi_L(\mathcal{L}_R^\epsilon \cap \mathcal{O}_\ell)$ such that $|\bar{x}(\mu_1)| = \mu_1\sigma_1$, $C^{-1}\sigma_1 \leq \mu_1 \leq C\sigma_1$. For μ_1 fixed, let $\{x(\mu_1, \mu_2)\}$ be a collection of σ_1 -separated points on the circle of radius $\mu_1\sigma_1$ centered at the origin, such that the angle between $x(\mu_1, \mu_2)$ and $\bar{x}(\mu_1)$ is $\mu_2\sigma_1$, $0 \leq \mu_2 \leq 2^\ell$. If $\mathbf{U}_{\mu_1, \mu_2}$ denotes a square of sidelength σ_1 centered at $x(\mu_1, \mu_2)$, then the squares $\{\mathbf{U}_{\mu_1, \mu_2}\}$ are essentially disjoint and there exists a constant $C > 0$ such that $\pi_L\mathcal{O}_\ell = \bigcup_{\mu_1, \mu_2} C\mathbf{U}_{\mu_1, \mu_2}$. The number of 2-tuples (μ_1, μ_2) needed for the covering is at most $C2^{j+\ell}$. We use μ_3 to index the cubes \mathbf{Q} whose centers lie in $\pi_L^{-1}(C\mathbf{U}_{\mu_1, \mu_2}) \cap \mathcal{O}_\ell$. By Lemma 6.7, the number of indices μ_3 corresponding to a given tuple (μ_1, μ_2) is bounded by $C2^\ell$. By throwing out the spurious indices, we can avoid overcounting, so that each cube \mathbf{Q} has a unique index μ .

It is obvious that there is a bijection between the sets of indices μ and ν . By a slight abuse of notation, we will sometimes denote a cube \mathbf{Q} by $\mathbf{Q}(\nu)$ or $\mathbf{Q}(\mu)$, the enumeration scheme being clear from the context. In fact, we will use the first scheme in the proof of (6.7), and the second in the proof of (6.8). The diagrams below depict the two enumeration schemes and properties of the projections π_L and π_R as outlined in Lemmas 6.5, 6.6 and 6.7.

x projections for fixed z near Γ_R^ε



z projections for fixed x near L_R^ε



Finally, we use the two enumeration schemes described above to decompose \mathcal{Q} into a finite number of subcollections \mathcal{Q}_i , as mentioned in §§6.2. If both P and R are sign-definite, then no decompositions are necessary and $N = 1$. If P is sign-indefinite, then for every $z \in \mathbb{R}_z^2$, γ_z^ε is a hyperbola centered at $-P^{-1}Qz$, with asymptotes along the directions $\pm p^{(1)}$ and $\pm p^{(2)}$, where

$$p^{(i)\dagger} P p^{(i)} = 0, \quad \|p^{(i)}\| = 1, \quad i = 1, 2.$$

We decompose the hyperbola γ_z^ε into four pieces, namely $\gamma_z^{\varepsilon, \pm 1}$ and $\gamma_z^{\varepsilon, \pm 2}$, where $\gamma_z^{\varepsilon, \pm i}$ is a connected segment of γ_z^ε asymptotic only to $\pm p^{(i)}$. We know from Lemma 6.6 that for every fixed (ν_1, ν_2) , $\cup_{\nu_3} \pi_L \mathbf{Q}(\nu_1, \nu_2, \nu_3)$ is contained in a $C2^{\ell-j-k}$ -long and $C\sigma_1$ -thick tubular neighborhood of $\gamma_{z(\nu_1, \nu_2)}^\varepsilon$. It is therefore possible to decompose \mathcal{Q} into four subcollections \mathcal{Q}_i^\pm , $i = 1, 2$, satisfying the following property: for every (ν_1, ν_2) , $\cup \{\pi_L(\mathbf{Q}); \mathbf{Q} = \mathbf{Q}(\nu_1, \nu_2, \nu_3) \in \mathcal{Q}_i^\pm\}$ is contained in a $C2^{\ell-j-k}$ -long and $C\sigma_1$ -thick tubular neighborhood of $\gamma_{z(\nu_1, \nu_2)}^{\varepsilon, \pm i}$. If R is sign-indefinite, we similarly define $r^{(i')}$, $i' = 1, 2$ (the “null” directions of R) and ${}_x \gamma^{\varepsilon, \pm i'}$ (pieces of ${}_x \gamma^\varepsilon$), and do a further subdivision of each \mathcal{Q}_i^\pm into $\mathcal{Q}_{i, i'}^{\pm, \pm}$, $i' = 1, 2$, to ensure that for every fixed (μ_1, μ_2) , the set $\cup \{\pi_R(\mathbf{Q}) : \mathbf{Q} = \mathbf{Q}(\mu_1, \mu_2, \mu_3) \in \mathcal{Q}_{i, i'}^{\pm, \pm}\}$ is contained in a $C2^\ell \sigma_0$ -long and $C\sigma_1$ -thick tubular neighborhood of ${}_{x(\mu_1, \mu_2)} \gamma^{\varepsilon, \pm i'}$. In what follows, the subcollection of \mathcal{Q} will always be fixed, and we will continue to denote by ${}_x \gamma^\varepsilon$ and γ_z^ε the segments of the respective curves that correspond to that subcollection.

7. PROOF OF PROPOSITION 6.4

7.1. A generalized Operator Van der Corput Lemma. We bound the L^2 -norm of the operator $\mathcal{T}_{\mathbf{Q}}\mathcal{T}_{\mathbf{Q}'}^*$ via the following standard estimate :

$$(7.1) \quad \|\mathcal{T}_{\mathbf{Q}}\mathcal{T}_{\mathbf{Q}'}^*\| \leq C \left[\sup_y \int |K_{\mathcal{T}_{\mathbf{Q}}\mathcal{T}_{\mathbf{Q}'}^*}(x, y)| dx \right]^{\frac{1}{2}} \left[\sup_x \int |K_{\mathcal{T}_{\mathbf{Q}}\mathcal{T}_{\mathbf{Q}'}^*}(x, y)| dy \right]^{\frac{1}{2}},$$

where $K_{\mathcal{T}_{\mathbf{Q}}\mathcal{T}_{\mathbf{Q}'}^*}$ is the Schwartz kernel of the $\mathcal{T}_{\mathbf{Q}}\mathcal{T}_{\mathbf{Q}'}^*$, given by

$$(7.2) \quad K_{\mathcal{T}_{\mathbf{Q}}\mathcal{T}_{\mathbf{Q}'}^*}(x, y) = \int e^{i\lambda[S(x, z) - S(y, z)]} b_{\mathbf{Q}}(x, z) \bar{b}_{\mathbf{Q}'}(y, z) dz.$$

Similar expressions hold for $\|\mathcal{T}_{\mathbf{Q}'}^*\mathcal{T}_{\mathbf{Q}}\|$ and $K_{\mathcal{T}_{\mathbf{Q}'}^*\mathcal{T}_{\mathbf{Q}}}$. The main ingredient in estimating the kernels $K_{\mathcal{T}_{\mathbf{Q}}\mathcal{T}_{\mathbf{Q}'}^*}$ and $K_{\mathcal{T}_{\mathbf{Q}'}^*\mathcal{T}_{\mathbf{Q}}}$ is the following generalization of the operator Van der Corput lemma and Young's inequality.

Lemma 7.1. Fix $\sigma_2 = c2^{-k}$, $\sigma_1 = c2^{-j-k}$, and $0 < \tau \leq \sigma_1$. Suppose $\mathbf{Q}, \mathbf{Q}' \in \mathcal{Q}$ are σ_1 -cubes such that $\pi_R \mathbf{Q}, \pi_R \mathbf{Q}' \subseteq \mathbf{R}$ for some $C\sigma_1 \times C\tau$ rectangle \mathbf{R} in \mathbb{R}_z^2 . Let

$$\mathcal{A}(\mathbf{Q}, \mathbf{Q}') = \{(x, y) : \text{there exists } z \in \mathbf{R} \text{ such that } (x, z) \in \mathbf{Q}, (y, z) \in \mathbf{Q}'\}.$$

Then for $c > 0$ sufficiently small, there exists an orthogonal matrix U_0 depending on \mathbf{Q}, \mathbf{Q}' such that for all $N \geq 1$,

$$(7.3) \quad |K_{\mathcal{T}_{\mathbf{Q}}\mathcal{T}_{\mathbf{Q}'}^*}(x, y)| \leq \frac{C_N \sigma_1 \tau}{(1 + \lambda \sigma_1^2 |u_1 - v_1|)^N (1 + \lambda \sigma_2 \sigma_1 |u_2 - v_2|)^N}$$

for $(x, y) \in \mathcal{A}(\mathbf{Q}, \mathbf{Q}')$, and $K_{\mathcal{T}_{\mathbf{Q}}\mathcal{T}_{\mathbf{Q}'}^*}(x, y) = 0$ otherwise. Here $u = U_0 x$ and $v = U_0 y$.

An analogous statement holds for $K_{\mathcal{T}_{\mathbf{Q}'}^*\mathcal{T}_{\mathbf{Q}}}$.

Proof. The integral in (7.2) is estimated using integration by parts. Setting $(\alpha_0, \gamma_0) = c(\mathbf{Q})$ and $\beta_0 = c_x(\mathbf{Q}')$, we compute

$$\begin{aligned} S'_z(x, z) - S'_z(y, z) &= \int_0^1 \frac{d}{dt} S'_z(tx + (1-t)y, z) dt \\ &= (x-y)^t \int_0^1 S''_{xz}(tx + (1-t)y, z) dt \\ &= (x-y)^t [A_0 + \mathcal{E}(x, y, z)], \end{aligned}$$

where

$$A_0 = A_0(\mathbf{Q}, \mathbf{Q}') = \int_0^1 S''_{xz}(t\alpha_0 + (1-t)\beta_0, \gamma_0) dt, \text{ and}$$

$$\mathcal{E} = \iint_{[0,1]^2} \frac{d}{ds} [S''_{xz}(s(tx + (1-t)y) + (1-s)(t\alpha_0 + (1-t)\beta_0), sz + (1-s)\gamma_0)] ds dt.$$

Since $z, \gamma_0 \in \mathbf{R}$, $(x, z) \in \mathbf{Q}$ and $(y, z) \in \mathbf{Q}'$, it follows that

$$\|\mathcal{E}\| \leq \|S\|_{C^3} (|x - \alpha_0| + |y - \beta_0| + |z - \gamma_0|) \leq C\sigma_1.$$

Let $A_0 = U_0^t D_0 V_0$ be the singular value decomposition of A_0 , where U_0, V_0 are orthogonal matrices, and D_0 is diagonal, with diagonal entries (d_1, d_2) . Then $|d_1| \sim 2^{-j-k}$, $|d_2| \sim 2^{-k}$. We define $\mathcal{E}'(x, y, z) = U_0 \mathcal{E} V_0^*$, and new variables

$$u = U_0 x, \quad v = U_0 y, \quad \text{and} \quad w = (I + D_0^{-1} \mathcal{E}') V_0 z.$$

Notice that if the constant c in the definition of σ_1 is chosen sufficiently small, then $z \mapsto w$ is an invertible transformation, and

$$\left| \frac{d}{dw_i} [S(x, z) - S(y, z)] \right| = |d_i| |u_i - v_i| \gtrsim \sigma_i |u_i - v_i|, \quad i = 1, 2.$$

Integrating the kernel (7.2) by parts N times in w_1 and w_2 , applying (6.6) and using the size of \mathbf{R} , we obtain the desired conclusion. \square

7.2. Proof of (6.8). In order to prove (6.8), we index the cubes in \mathcal{Q} by the second scheme outlined in subsection 6.3 and observe from Lemma 7.1 that $K_{\mathcal{T}_{\mathbf{Q}}^* \mathcal{T}_{\mathbf{Q}'}} = 0$ for $\mathbf{Q} = \mathbf{Q}(\mu)$, $\mathbf{Q}' = \mathbf{Q}(\mu')$ if $|\mu_1 - \mu'_1| + |\mu_2 - \mu'_2| \geq C$ for some large constant C . We can therefore assume that $|\mu - \mu'| \sim |\mu_3 - \mu'_3|$. By Lemma 6.5, both $\pi_L \mathbf{Q}, \pi_L \mathbf{Q}' \subseteq C\mathbf{U}$ for some square \mathbf{U} in \mathbb{R}_x^2 of sidelength σ_1 . Using Lemma 7.1 (with the roles of x and z interchanged, \mathbf{R} replaced by \mathbf{U} and $\tau = \sigma_1$) we obtain an orthogonal matrix V_0 such that for $(z, w) \in \tilde{\mathcal{A}}(\mathbf{Q}, \mathbf{Q}')$,

$$\left| K_{\mathcal{T}_{\mathbf{Q}}^* \mathcal{T}_{\mathbf{Q}'}}(z, w) \right| \leq \frac{C_N \sigma_1^2}{(1 + \lambda \sigma_1^2 |s_1 - t_1|)^N (1 + \lambda \sigma_1 \sigma_2 |s_2 - t_2|)^N}, \quad s = V_0 z, t = V_0 w.$$

Here $\tilde{\mathcal{A}}(\mathbf{Q}, \mathbf{Q}') = \{(z, w) : \text{there exists } x \in \mathbf{U} \text{ such that } (x, z) \in \mathbf{Q}, (x, w) \in \mathbf{Q}'\}$. Let us decompose $C\mathbf{U}_{\mu_1, \mu_2}$ as follows

$$C\mathbf{U}_{\mu_1, \mu_2} = \bigcup_{\kappa} \mathbf{U}_{\mu_1, \mu_2}(\kappa),$$

where $\{\mathbf{U}_{\mu_1, \mu_2}(\kappa) : \kappa \lesssim 2^{j-\ell}\}$ is an essentially disjoint collection of subsets with the property that $\pi_R(\pi_L^{-1} \mathbf{U}_{\mu_1, \mu_2}(\kappa) \cap \mathcal{O}_\ell)$ is a σ_0 -thick tubular neighborhood of $x\gamma^\epsilon$ for some $x \in \mathbf{U}_{\mu_1, \mu_2}(\kappa)$. It then follows that

$$\begin{aligned} \tilde{\mathcal{A}}(\mathbf{Q}, \mathbf{Q}') &\subseteq \bigcup_{\kappa} \{(z, w) : \exists x \in \mathbf{U}_{\mu_1, \mu_2}(\kappa) \text{ with } (x, z) \in \mathbf{Q}, (x, w) \in \mathbf{Q}'\} \\ &\subseteq \bigcup_{\kappa} \tilde{\mathbf{R}}_{\kappa}(\mathbf{Q}) \times \tilde{\mathbf{R}}_{\kappa}(\mathbf{Q}'), \end{aligned}$$

where $\tilde{\mathbf{R}}_{\kappa}(\mathbf{Q})$ and $\tilde{\mathbf{R}}_{\kappa}(\mathbf{Q}')$ are $C\sigma_0$ -squares in \mathbb{R}_z^2 satisfying

$$\pi_R(\pi_L^{-1} \mathbf{U}_{\mu_1, \mu_2}(\kappa) \cap \mathbf{Q}) \subseteq \tilde{\mathbf{R}}_{\kappa}(\mathbf{Q}), \quad \pi_R(\pi_L^{-1} \mathbf{U}_{\mu_1, \mu_2}(\kappa) \cap \mathbf{Q}') \subseteq \tilde{\mathbf{R}}_{\kappa}(\mathbf{Q}').$$

Since both $z, w \in x\gamma^\epsilon$ for some $x \in \mathbf{U}_{\mu_1, \mu_2}(\kappa)$, the length of the curve $x\gamma^\epsilon$ between z and w is $\sim n\sigma_0$, and hence $|z - w| = |s - t| \sim n\sigma_0$, where $n = |\mu_3 - \mu'_3|$. Further, since $K_{\mathcal{T}_{\mathbf{Q}}^* \mathcal{T}_{\mathbf{Q}'}}$ is symmetric in s and t , in order to compute $\|\mathcal{T}_{\mathbf{Q}}^* \mathcal{T}_{\mathbf{Q}'}\|$, it suffices to only estimate

$$\begin{aligned} \sup_w \int \left| K_{\mathcal{T}_{\mathbf{Q}}^* \mathcal{T}_{\mathbf{Q}'}}(z, w) \right| dz &\leq \sup_{\kappa} \sup_{w \in \tilde{\mathbf{R}}_{\kappa}(\mathbf{Q}')} \int_{\tilde{\mathbf{R}}_{\kappa}(\mathbf{Q})} \left| K_{\mathcal{T}_{\mathbf{Q}}^* \mathcal{T}_{\mathbf{Q}'}}(z, w) \right| dz \\ &\leq \sup_{\kappa} \sup_{t \in V_0 \tilde{\mathbf{R}}_{\kappa}(\mathbf{Q}')} \left(\tilde{\mathcal{I}}_1 + \tilde{\mathcal{I}}_2 \right), \end{aligned}$$

where for $i = 1, 2$ and $t \in V_0 \tilde{\mathbf{R}}_{\kappa}(\mathbf{Q}')$,

$$\begin{aligned} \tilde{\mathcal{I}}_i &= \tilde{\mathcal{I}}_i(t, \kappa, n) = \iint_{\mathcal{S}_i} \frac{C_N \sigma_1^2 ds}{\prod_{r=1}^2 (1 + \lambda \sigma_1 \sigma_r |s_r - t_r|)^N}, \text{ and} \\ \mathcal{S}_i &= \mathcal{S}_i(t) = \{s : |s - s(\mathbf{Q}, \mu)| \leq C\sigma_0, |s_i - t_i| \geq n\sigma_0/2\}. \end{aligned}$$

Here $s(\mathbf{Q}, \kappa) = V_0 z(\mathbf{Q}, \kappa)$, where $z(\mathbf{Q}, \kappa)$ is the center of $\tilde{\mathbf{R}}_\kappa(\mathbf{Q})$.

We show that

$$\sum_{\ell, j, k} \sum_{n \lesssim 2^\ell} \sqrt{\sup_{\kappa, t} \tilde{\mathcal{I}}_1} \lesssim \lambda^{-\frac{2}{3}},$$

the proof for $\tilde{\mathcal{I}}_2$ being similar and left to the reader. By part (c) of Lemma 6.7,

$$(7.4) \quad |s_2 - t_2| \gtrsim n^2 \sigma_0^2 \sigma_2^{-1} \text{ for } s \in \mathcal{S}_1.$$

Therefore,

$$\tilde{\mathcal{I}}_1 \leq \frac{C_N \sigma_1^2}{(1 + \lambda \sigma_1^2 \sigma_0)^N (1 + \lambda \sigma_1 \sigma_0^2 n^2)^N} \min\left(\sigma_0, \frac{1}{\lambda \sigma_1^2}\right) \min\left(\sigma_0, \frac{1}{\lambda \sigma_1 \sigma_2}\right).$$

Summing over $n \lesssim 2^\ell$ we obtain

$$\sum_{n \lesssim 2^\ell} \sqrt{\sup_{\kappa, t} \tilde{\mathcal{I}}_1} \lesssim \sigma_1 \min\left(\frac{1}{\lambda \sigma_1^2 \sigma_0}, \frac{1}{\sqrt{\lambda \sigma_1 \sigma_0^2}}, 2^\ell\right) \min\left(\sigma_0, \frac{1}{\lambda \sigma_1^2}\right)^{\frac{1}{2}} \min\left(\sigma_0, \frac{1}{\lambda \sigma_1 \sigma_2}\right)^{\frac{1}{2}}.$$

The following cases arise:

Case 1 : Suppose $\lambda \sigma_1^3 > 1$, i.e., $\lambda 2^{-3j-3k} > 1$. This in particular implies that $1/(\lambda \sigma_1^2 \sigma_0) < 1/\sqrt{\lambda \sigma_1 \sigma_0^2}$. Therefore,

$$\begin{aligned} \sum_{n \lesssim 2^\ell} \sqrt{\sup_{\kappa, t} \tilde{\mathcal{I}}_1} &\lesssim \sigma_1 \min\left(\frac{1}{\lambda \sigma_1^2 \sigma_0}, 2^\ell\right) \left(\frac{1}{\lambda \sigma_1 \sigma_2}\right)^{\frac{1}{2}} \min\left(\sigma_0, \frac{1}{\lambda \sigma_1^2}\right)^{\frac{1}{2}} \\ &\lesssim \begin{cases} \sigma_1 \frac{1}{\lambda \sigma_1^2 \sigma_0} \left(\frac{1}{\lambda \sigma_1 \sigma_2}\right)^{\frac{1}{2}} \left(\frac{1}{\lambda \sigma_1^2}\right)^{\frac{1}{2}} & \text{if } \sigma_0 \geq \frac{1}{\lambda \sigma_1^2}; \\ \sigma_1 \frac{1}{\lambda \sigma_1^2 \sigma_0} \left(\frac{1}{\lambda \sigma_1 \sigma_2}\right)^{\frac{1}{2}} \sigma_0^{\frac{1}{2}} & \text{if } \sigma_0 < \frac{1}{\lambda \sigma_1^2}, \frac{1}{\lambda \sigma_1 \sigma_2} < 2^\ell; \\ \sigma_1 2^\ell \left(\frac{1}{\lambda \sigma_1 \sigma_2}\right)^{\frac{1}{2}} \sigma_0^{\frac{1}{2}} & \text{if } 2^\ell \leq \frac{1}{\lambda \sigma_1^2 \sigma_0}. \end{cases} \end{aligned}$$

Subcase 1 : Suppose $\sigma_0 \geq 1/(\lambda \sigma_1^2)$, i.e., $2^\ell \geq \lambda^{-1} 2^{4j+3k}$. Therefore,

$$\begin{aligned} \sum_{\ell, j, k} \sum_n \sqrt{\sup_{\kappa, t} \tilde{\mathcal{I}}_1} &\lesssim \sum_{\ell, j, k} \sigma_1 \frac{1}{\lambda \sigma_1^2 \sigma_0} \left(\frac{1}{\lambda \sigma_1 \sigma_2}\right)^{\frac{1}{2}} \left(\frac{1}{\lambda \sigma_1^2}\right)^{\frac{1}{2}} \\ &\lesssim \sum_{\ell, j, k} \lambda^{-2} \sigma_1^{-\frac{5}{2}} \sigma_0^{-1} \sigma_2^{-\frac{1}{2}} \\ &\lesssim \sum_{j, k} \sum_{2^\ell > \lambda^{-1} 2^{4j+3k}} \lambda^{-2} 2^{-\ell + \frac{9j}{2} + 4k} \\ &\lesssim \sum_k \sum_{2^{3j+3k} \leq \lambda} \lambda^{-1} 2^{\frac{j}{2} + k} \\ &\lesssim \lambda^{-1} \sum_{\lambda 2^{-3k} \geq 1} (\lambda 2^{-3k})^{\frac{1}{6}} 2^k \\ &\lesssim \lambda^{-1 + \frac{1}{6} + \frac{1}{6}} = \lambda^{-\frac{2}{3}}. \end{aligned}$$

Subcase 2 : Suppose that $\sigma_0 < 1/(\lambda\sigma_1^2)$ and $2^\ell > 1/(\lambda\sigma_1^2\sigma_0)$. The second inequality is equivalent to $2^\ell > \lambda^{-\frac{1}{2}}2^{2j+\frac{3k}{2}}$. The summation then yields

$$\begin{aligned} \sum_{\ell,j,k} \sum_n \sqrt{\sup_{\kappa,t} \tilde{\mathcal{I}}_1} &\lesssim \sum_{\ell,j,k} \sigma_1 \frac{1}{\lambda\sigma_1^2\sigma_0} \left(\frac{1}{\lambda\sigma_1\sigma_2} \right)^{\frac{1}{2}} \sigma_0^{\frac{1}{2}} \\ &\lesssim \sum_{j,k} \sum_{2^\ell > \lambda^{-\frac{1}{2}}2^{2j+\frac{3k}{2}}} \lambda^{-\frac{3}{2}} 2^{\frac{5j+5k}{2}} 2^{-\frac{\ell}{2}} \\ &\lesssim \sum_k \sum_{2^j < (\lambda 2^{-3k})^{\frac{1}{3}}} \lambda^{-\frac{3}{2}+\frac{1}{4}} 2^{\frac{3j}{2}+\frac{7k}{4}} \\ &\lesssim \lambda^{-\frac{3}{2}+\frac{1}{4}} \sum_{\lambda 2^{-3k} \geq 1} (\lambda 2^{-3k})^{\frac{1}{2}} 2^{\frac{7k}{4}} \\ &\lesssim \lambda^{-\frac{3}{4}} \sum_{2^k < \lambda^{\frac{1}{3}}} 2^{\frac{k}{4}} \lesssim \lambda^{-\frac{2}{3}}. \end{aligned}$$

Subcase 3 : If $2^\ell \leq 1/(\lambda\sigma_1^2\sigma_0)$, i.e., $2^\ell \leq \lambda^{-\frac{1}{2}}2^{2j+\frac{3k}{2}}$, then

$$\begin{aligned} \sum_{\ell,j,k} \sum_n \sqrt{\sup_{\kappa,t} \tilde{\mathcal{I}}_1} &\lesssim \sum_{\ell,j,k} \sigma_1 2^\ell \left(\frac{1}{\lambda\sigma_1\sigma_2} \right)^{\frac{1}{2}} \sigma_0^{\frac{1}{2}} \\ &\lesssim \sum_{j,k} \sum_{2^\ell \leq \lambda^{-\frac{1}{2}}2^{2j+\frac{3k}{2}}} \lambda^{-\frac{1}{2}} 2^{\frac{3\ell}{2}-\frac{3j}{2}-\frac{k}{2}} \\ &\lesssim \lambda^{-\frac{1}{2}-\frac{3}{4}} \sum_k \sum_{2^j \leq (\lambda 2^{-3k})^{\frac{1}{3}}} 2^{\frac{3j}{2}+\frac{7k}{4}} \\ &\lesssim \lambda^{-\frac{1}{2}-\frac{3}{4}} \sum_{\lambda 2^{-3k} \geq 1} (\lambda 2^{-3k})^{\frac{1}{2}} 2^{\frac{7k}{4}} \lesssim \lambda^{-\frac{2}{3}}. \end{aligned}$$

Case 2 : Suppose $\lambda\sigma_1^3 \leq 1$ and $\lambda\sigma_0\sigma_1\sigma_2 \leq 1$, i.e.,

$$(7.5) \quad 2^{3j+3k} \geq \lambda \quad \text{and} \quad 2^\ell \leq \min(2^j, \lambda^{-1}2^{3j+3k}).$$

Then

$$\begin{aligned} \sum_{n \lesssim 2^\ell} \sqrt{\sup_{\kappa,t} \tilde{\mathcal{I}}_1} &\lesssim \sigma_1\sigma_0 \min\left(\frac{1}{\sqrt{\lambda\sigma_1\sigma_0^2}}, 2^\ell \right) \\ &\lesssim \begin{cases} \sigma_1\sigma_0 2^\ell & \text{if } 2^\ell \leq \frac{1}{\sqrt{\lambda\sigma_1\sigma_0^2}}; \\ \sigma_1\sigma_0 \frac{1}{\sqrt{\lambda\sigma_1\sigma_0^2}} & \text{if } 2^\ell > \frac{1}{\sqrt{\lambda\sigma_1\sigma_0^2}} \end{cases}. \end{aligned}$$

Subcase 1 : Let $2^\ell \leq 1/\sqrt{\lambda\sigma_1\sigma_0^2}$, i.e., $2^\ell \leq \lambda^{-\frac{1}{4}}2^{\frac{5j+3k}{4}}$. Combining this with (7.5), we obtain the following range of ℓ ,

$$(7.6) \quad 2^\ell \leq \min\left(2^j, \lambda^{-1}2^{3j+3k}, \lambda^{-\frac{1}{4}}2^{\frac{5j+3k}{4}} \right).$$

If the minimum in (7.6) is 2^j , then in particular $2^j \leq \lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}}$, which implies $2^j \geq \lambda 2^{-3k}$. This means that

$$\begin{aligned} \sum_{\ell,j,k} \sum_n \sqrt{\sup_{\kappa,t} \tilde{\mathcal{I}}_1} &\lesssim \sum_{\ell,j,k} \sigma_1 \sigma_0 2^\ell \\ &\lesssim \sum_{j,k} \sum_{\ell \leq j} 2^{2\ell-3j-2k} \\ &\lesssim \sum_k \sum_{2^j \geq \lambda 2^{-3k}} 2^{-j-2k} \\ &\lesssim \lambda^{-1} \sum_{\lambda 2^{-3k} > 1} 2^k + \sum_{\lambda 2^{-3k} \leq 1} 2^{-2k} \\ &\lesssim \lambda^{-\frac{2}{3}}. \end{aligned}$$

If the minimum in (7.6) is $\lambda^{-1} 2^{3j+3k}$, then $\lambda^{-1} 2^{3j+3k} \leq \lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}}$, i.e., $2^j \leq (\lambda 2^{-3k})^{\frac{3}{7}}$. In this case,

$$\begin{aligned} \sum_{\ell,j,k} \sum_n \sqrt{\sup_{\kappa,t} \tilde{\mathcal{I}}_1} &\lesssim \sum_{j,k} \sum_{2^\ell \leq \lambda^{-1} 2^{3j+3k}} 2^{2\ell-3j-3k} \\ &\lesssim \lambda^{-2} \sum_k \sum_{2^j \leq (\lambda 2^{-3k})^{\frac{3}{7}}} 2^{3j+4k} \\ &\lesssim \lambda^{-2} \sum_{\lambda 2^{-3k} \geq 1} (\lambda 2^{-3k})^{\frac{9}{7}} 2^{4k} \\ &\lesssim \lambda^{-\frac{5}{7}} \sum_{2^k \leq \lambda^{\frac{1}{3}}} 2^{\frac{k}{7}} \lesssim \lambda^{-\frac{2}{3}}. \end{aligned}$$

If the minimum in (7.6) is $\lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}}$, then $\lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}} \leq \lambda^{-1} 2^{3j+3k}$, i.e., $2^j \geq (\lambda 2^{-3k})^{\frac{3}{7}}$. The summation now gives,

$$\begin{aligned} \sum_{\ell,j,k} \sum_n \sqrt{\sup_{\kappa,t} \tilde{\mathcal{I}}_1} &\lesssim \sum_{j,k} \sum_{2^\ell \leq \lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}}} 2^{2\ell-3j-2k} \\ &\lesssim \sum_k \sum_{2^j \geq (\lambda 2^{-3k})^{\frac{3}{7}}} \lambda^{-\frac{1}{2}} 2^{-\frac{j+k}{2}} \\ &\lesssim \lambda^{-\frac{1}{2}} \sum_{\lambda 2^{-3k} \geq 1} (\lambda 2^{-3k})^{-\frac{3}{14}} 2^{-\frac{k}{2}} + \lambda^{-\frac{1}{2}} \sum_{\lambda 2^{-3k} < 1} 2^{-\frac{k}{2}} \\ &\lesssim \lambda^{-\frac{1}{2}-\frac{3}{14}} \sum_{2^k \leq \lambda^{\frac{1}{3}}} 2^{\frac{k}{7}} + \lambda^{-\frac{1}{2}-\frac{1}{6}} \lesssim \lambda^{-\frac{2}{3}}. \end{aligned}$$

Subcase 2 : Let $2^\ell > 1/\sqrt{\lambda \sigma_1 \sigma_0^2}$, i.e., $2^\ell > \lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}}$.

If the upper bound for ℓ given in (7.5) is j , i.e., $2^j \leq \lambda^{-1} 2^{3j+3k}$ or $2^j \geq (\lambda 2^{-3k})^{\frac{1}{2}}$, then

$$\sum_{\ell,j,k} \sum_n \sqrt{\sup_{\kappa,t} \tilde{\mathcal{I}}_1} \lesssim \sum_{\ell,j,k} \lambda^{-\frac{1}{2}} \sigma_1^{\frac{1}{2}}$$

$$\begin{aligned}
 &\lesssim \sum_{j,k} \lambda^{-\frac{1}{2}} 2^{-\frac{j+k}{2}} \min [j, \log (\lambda^{-1} 2^{3j+3k})] \\
 &\lesssim \sum_k \sum_{2^j \geq (\lambda 2^{-3k})^{\frac{1}{2}}} \lambda^{-\frac{1}{2}} j 2^{-\frac{j+k}{2}} \\
 &\lesssim \lambda^{-\frac{1}{2}} \sum_{\lambda 2^{-3k} \geq 1} \log (\lambda 2^{-3k}) (\lambda 2^{-3k})^{-\frac{1}{4}} 2^{-\frac{k}{2}} + \lambda^{-\frac{1}{2}} \sum_{\lambda 2^{-3k} \leq 1} 2^{-\frac{k}{2}} \\
 &\lesssim \lambda^{-\frac{2}{3}}.
 \end{aligned}$$

Suppose next that the upper bound for 2^ℓ given in (7.5) is $\lambda^{-1} 2^{3j+3k}$. A consequence of this is:

$$\lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}} \leq \lambda^{-1} 2^{3j+3k} \quad \text{or} \quad (\lambda 2^{-3k})^{\frac{3}{7}} \leq 2^j \leq (\lambda 2^{-3k})^{\frac{1}{2}},$$

which leads to

$$\begin{aligned}
 \sum_{\ell,j,k} \sum_n \sqrt{\sup_{\kappa,t} \tilde{\mathcal{I}}_1} &\lesssim \sum_k \sum_{2^j \geq (\lambda 2^{-3k})^{\frac{3}{7}}} \lambda^{-\frac{1}{2}} 2^{-\frac{j+k}{2}} \log (\lambda^{-1} 2^{3j+3k}) \\
 &\lesssim \sum_{\lambda 2^{-3k} > 1} \lambda^{-\frac{1}{2}} \log \left(\lambda^{-1} 2^{3k} (\lambda 2^{-3k})^{\frac{9}{7}} \right) (\lambda 2^{-3k})^{-\frac{3}{14}} 2^{-\frac{k}{2}} \\
 &\lesssim \lambda^{-\frac{1}{2} - \frac{3}{14}} \sum_{2^k < \lambda^{\frac{1}{3}}} \log (\lambda 2^{-3k}) 2^{\frac{k}{7}} \lesssim \lambda^{-\frac{2}{3}}.
 \end{aligned}$$

Case 3 : Suppose $\lambda \sigma_1^3 \leq 1$ and $\lambda \sigma_0 \sigma_1 \sigma_2 \geq 1$. This is equivalent to $2^\ell \geq \lambda^{-1} 2^{3j+3k} \geq 1$. Then

$$\begin{aligned}
 \sum_n \sqrt{\sup_{\kappa,t} \tilde{\mathcal{I}}_1} &\lesssim \sigma_1 \min \left(\frac{1}{\sqrt{\lambda \sigma_1 \sigma_0^2}}, 2^\ell \right) \sigma_0^{\frac{1}{2}} \frac{1}{\sqrt{\lambda \sigma_1 \sigma_2}} \\
 &\lesssim \begin{cases} \lambda^{-1} \sigma_0^{-\frac{1}{2}} \sigma_2^{-\frac{1}{2}} & \text{if } \frac{1}{\sqrt{\lambda \sigma_1 \sigma_0^2}} \leq 2^\ell \\ \lambda^{-\frac{1}{2}} 2^\ell \sigma_1^{\frac{1}{2}} \sigma_0^{\frac{1}{2}} \sigma_2^{-\frac{1}{2}} & \text{if } \frac{1}{\sqrt{\lambda \sigma_1 \sigma_0^2}} > 2^\ell \end{cases}.
 \end{aligned}$$

Subcase 1 : Let $2^\ell \geq 1/\sqrt{\lambda \sigma_1 \sigma_0^2}$, or $2^\ell \geq \lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}}$. Therefore,

$$(7.7) \quad 2^\ell \geq \max \left(\lambda^{-1} 2^{3j+3k}, \lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}} \right).$$

If the maximum in (7.7) is $\lambda^{-1} 2^{3j+3k}$, i.e.,

$$\lambda^{-1} 2^{3j+3k} \geq \lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}} \quad \text{or} \quad 2^j \geq (\lambda 2^{-3k})^{\frac{3}{7}},$$

then

$$\begin{aligned}
\sum_{\ell,j,k} \sum_n \sqrt{\sup_{\kappa,t} \tilde{\mathcal{I}}_1} &\lesssim \sum_{\ell,j,k} \lambda^{-1} \sigma_0^{-\frac{1}{2}} \sigma_2^{-\frac{1}{2}} \\
&\lesssim \sum_{j,k} \sum_{2^\ell \geq \lambda^{-1} 2^{3j+3k}} \lambda^{-1} 2^{-\frac{\ell}{2}+j+k} \\
&\lesssim \sum_k \sum_{2^j \geq (\lambda 2^{-3k})^{\frac{3}{7}}} \lambda^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} 2^{-\frac{j+k}{2}} \\
&\lesssim \lambda^{-\frac{1}{2}} \sum_{\lambda 2^{-3k} \geq 1} 2^{-\frac{k}{2}} (\lambda 2^{-3k})^{-\frac{3}{14}} + \sum_{\lambda 2^{-3k} \leq 1} 2^{-\frac{k}{2}} \lambda^{-\frac{1}{2}} \\
&\lesssim \lambda^{-\frac{1}{2}-\frac{3}{14}} \sum_{2^k < \lambda^{\frac{1}{3}}} 2^{\frac{k}{7}} + \lambda^{-\frac{1}{2}-\frac{1}{6}} \\
&\lesssim \lambda^{-\frac{2}{3}}.
\end{aligned}$$

Subcase 2 : Suppose $2^\ell < 1/\sqrt{\lambda \sigma_1 \sigma_0^2}$, i.e., $2^\ell < \lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}}$. Since $2^\ell > \lambda^{-1} 2^{3j+3k}$, and $\ell \leq j$, therefore combining the above statements we obtain $2^j < (\lambda 2^{-3k})^{\frac{3}{7}}$. The summation then proceeds as follows,

$$\begin{aligned}
\sum_{\ell,j,k} \sum_n \sqrt{\sup_{\kappa,t} \tilde{\mathcal{I}}_1} &\lesssim \sum_{\ell,j,k} \lambda^{-\frac{1}{2}} 2^\ell \sigma_1^{\frac{1}{2}} \sigma_0^{\frac{1}{2}} \sigma_2^{-\frac{1}{2}} \\
&\lesssim \sum_{j,k} \sum_{2^\ell < \lambda^{-\frac{1}{4}} 2^{\frac{5j+3k}{4}}} \lambda^{-\frac{1}{2}} 2^{\frac{3\ell-3j-k}{2}} \\
&\lesssim \sum_k \sum_{2^j < (\lambda 2^{-3k})^{\frac{3}{7}}} \lambda^{-\frac{1}{2}-\frac{3}{8}} 2^{\frac{3j+5k}{8}} \\
&\lesssim \lambda^{-\frac{1}{2}-\frac{3}{8}} \sum_{2^k < \lambda^{\frac{1}{3}}} (\lambda 2^{-3k})^{\frac{9}{56}} 2^{\frac{5k}{8}} \lesssim \lambda^{-\frac{2}{3}}.
\end{aligned}$$

□

7.3. Proof of (6.7). We now employ the first enumeration scheme for indexing the cubes in \mathcal{Q} , as described in §§6.3. It follows from Lemma 7.1 that $K_{\mathcal{T}_{\mathbf{Q}} \mathcal{T}_{\mathbf{Q}'}} = 0$ for $\mathbf{Q} = \mathbf{Q}(\nu)$, $\mathbf{Q}' = \mathbf{Q}(\nu')$, if $|\nu_1 - \nu'_1| + |\nu_2 - \nu'_2| \geq C$. Let us assume therefore that $|\nu - \nu'| \sim |\nu_3 - \nu'_3|$. By Lemma 6.5, both $\pi_R(\mathbf{Q}), \pi_R(\mathbf{Q}') \subseteq C\mathbf{R}_{\nu_1, \nu_2}$, where $\mathbf{R}_{\nu_1, \nu_2}$ is the $\sigma_1 \times \sigma_0$ rectangle described in §§6.3. Using Lemma 7.1 with $\mathbf{R} = C\mathbf{R}_{\nu_1, \nu_2}$, and $\tau = \sigma_0$, we obtain an orthogonal matrix U_0 such that for $(x, y) \in \mathcal{A}(\mathbf{Q}, \mathbf{Q}')$,

$$\left| K_{\mathcal{T}_{\mathbf{Q}} \mathcal{T}_{\mathbf{Q}'}}(x, y) \right| \leq \frac{C_N \sigma_1 \sigma_0}{(1 + \lambda \sigma_1^2 |u_1 - v_1|)^N (1 + \lambda \sigma_1 \sigma_2 |u_2 - v_2|)^N}, \quad u = U_0 x, v = U_0 y.$$

Since $x, y \in \gamma_z^\epsilon$ for some $z \in C\mathbf{R}_{\nu_1, \nu_2}$, the length of the curve γ_z^ϵ between x and y is $\sim n\sigma_1$, and so $|x - y| = |u - v| \sim n\sigma_1$, where $n = |\nu_3 - \nu'_3|$. As in the proof of (6.8), we use the symmetry in u and v to deduce that

$$\|\mathcal{T}_{\mathbf{Q}} \mathcal{T}_{\mathbf{Q}'}^*\| \leq \sup_y \int \left| K_{\mathcal{T}_{\mathbf{Q}} \mathcal{T}_{\mathbf{Q}'}}(x, y) \right| dx.$$

However, the estimation of the kernel in this case does not exactly follow the treatment of (6.8). The reason for this is that unlike $x\gamma^\epsilon$, the curve γ_z^ϵ for $z \in C\mathbf{R}_{\nu_1, \nu_2}$

need not be well-curved, and in particular this means that we do not always have the lower bound on the curvature that led to (7.4). We explain this below in greater detail.

The equation for γ_z^ϵ is given by

$$(x + P^{-1}Qz)^t P(x + P^{-1}Qz) = \epsilon - z^t(R - Q^t P^{-1}Q)z.$$

For $z \in \mathbf{R}_{\nu_1, \nu_2}$, the angular separation between z and $\bar{z}(\nu_1)$ is $\nu_2 \sigma_0 \sigma_2^{-1}$, and by our choice $\bar{z}(\nu_1) \in \Gamma_R^\epsilon$. This implies that

$$|z^t(R - Q^t P^{-1}Q)z - \epsilon| \sim \nu_2 \sigma_0 \sigma_2^{-1} \sigma_2^2 = \nu_2 \sigma_0 \sigma_2.$$

If P is sign-definite, then γ_z^ϵ is an ellipse with curvature bounded below by a multiple of

$$(\nu_2 \sigma_0 \sigma_2)^{-\frac{1}{2}} \gtrsim 2^{j-\ell-k} \gtrsim \sigma_2^{-1}, \text{ since } \nu_2 \lesssim 2^\ell.$$

In this case the treatment of the kernel $K_{\mathcal{T}_Q \mathcal{T}_Q^*}$ is similar to the one outlined in the proof of (6.8), and we leave the verification of this to the reader.

If P is sign-indefinite, then γ_z^ϵ is a hyperbola. Let us denote by (x', z') the point in \mathcal{L}_R^ϵ closest to (x, z) . Since the distance of (x, z) from \mathcal{L}_R^ϵ is $\sim 2^{\ell-j-k}$, this implies that

$$(7.8) \quad |x + P^{-1}Qz| = |(x - x') + P^{-1}Q(z - z')| \sim 2^{\ell-j-k}.$$

The curvature of $\gamma_z^\epsilon \cap \mathcal{O}_\ell$ is therefore of the order of

$$\frac{|\epsilon - z^t(R - Q^t P^{-1}Q)z|}{|x + P^{-1}Qz|^3} \sim \frac{\nu_2 \sigma_2 \sigma_0}{(2^{\ell-j-k})^3} = \nu_2 2^{-2\ell+j+k}.$$

This gives rise to two possibilities. If $\nu_2 2^{-2\ell+j} \geq c > 0$ (for some small constant c to be determined in the sequel), then once again we can use the lower bound σ_2^{-1} of the curvature and summation techniques similar to the ones used in the proof of (6.8) to obtain the desired sum of $C\lambda^{-\frac{2}{3}}$.

We therefore concentrate only on the case $\nu_2 2^{-2\ell+j} \leq c'$, where curvature does not help any longer. The main ingredient of the proof here is following claim : for $\mathbf{Q} = \mathbf{Q}(\nu)$, $\mathbf{Q}' = \mathbf{Q}(\nu')$, $|\nu - \nu'| \sim n = |\nu_3 - \nu'_3|$, and U_0 as in Lemma 7.1,

$$(7.9) \quad |u_2 - v_2| \gtrsim 2^{\ell-j} n \sigma_1, \text{ where } u - v = U_0(x - y), (x, z) \in \mathbf{Q}, (y, z) \in \mathbf{Q}'.$$

In order to prove (7.9), let us denote by p the unit vector pointing in the direction of the (unique) asymptote of γ_z^ϵ . Two cases arise, depending on whether $(S''_{xz})^t(\cdot, \cdot)p$ vanishes on \mathcal{L}_R^0 or not. (Note that S''_{xz} is linear in its arguments, therefore if it vanishes at a point on a line passing through the origin, then it vanishes on the entire line).

First suppose that $(S''_{xz})^t(\cdot, \cdot)p$ is nonzero on \mathcal{L}_R^0 , say

$$(7.10) \quad (S''_{xz})^t(-P^{-1}Qz_0, z_0)p = 2c_0 \neq 0 \text{ for } z_0^t(R - Q^t P^{-1}Q)z_0 = 0, |z_0| = 1.$$

Recall the definition of the matrices A_0 and U_0 from Lemma 7.1. From the linearity of S''_{xz} it follows that if $(\alpha_0, \gamma_0) = c(\mathbf{Q})$ and $\beta_0 = c_x(\mathbf{Q}')$, then

$$\begin{aligned} A_0 &= A_0(\mathbf{Q}, \mathbf{Q}') = \int_0^1 S''_{xz}(t\alpha_0 + (1-t)\beta_0, \gamma_0) dt \\ &= \frac{1}{2} [S''_{xz}(\alpha_0, \gamma_0) + S''_{xz}(\beta_0, \gamma_0)]. \end{aligned}$$

Since $(\alpha_0, \gamma_0) \in \mathbf{Q}$, $(\beta_0, \gamma_0) \in C\mathbf{Q}'$, and $\mathbf{Q}, \mathbf{Q}' \subseteq \mathcal{O}_\ell$, there exist $(x_0(\epsilon), z_0(\epsilon))$, $(x'_0(\epsilon), z'_0(\epsilon)) \in \mathcal{L}_R^\epsilon$ such that

$$(7.11) \quad |(\alpha_0, \gamma_0) - (x_0(\epsilon), z_0(\epsilon))|, |(\beta_0, \gamma_0) - (x'_0(\epsilon), z'_0(\epsilon))| \sim 2^{\ell-j-k}.$$

Moreover, there exist $(x_0, z_0), (x'_0, z'_0) \in \mathcal{L}_R^0$ such that

$$(7.12) \quad |(x_0(\epsilon), z_0(\epsilon)) - (x_0, z_0)| + |(x'_0(\epsilon), z'_0(\epsilon)) - (x'_0, z'_0)| \lesssim 2^{-j-k}.$$

Therefore, if $j \geq C$ and $\ell - j \leq -C$ for some large constant C , then comparing $S''_{xz}(\alpha_0, \gamma_0)$ and $S''_{xz}(\beta_0, \gamma_0)$ with $S''_{xz}(x_0, z_0)$ and $S''_{xz}(x'_0, z'_0)$ respectively and applying (7.10) gives $2^k |A_0^t p| \geq |c_0| > 0$. Using the singular value decomposition of A_0 we obtain,

$$\begin{aligned} 0 \neq |c_0|^2 2^{-2k} &\leq |A_0^t p|^2 = p^t U_0^t D_0^2 U_0 p = |d_1|^2 |e_1^t U_0 p|^2 + |d_2|^2 |e_2^t U_0 p|^2 \\ &\lesssim 2^{-2j-2k} + 2^{-2k} |e_2^t U_0 p|^2, \end{aligned}$$

where $\{e_1, e_2\}$ is the canonical basis of \mathbb{R}^2 . For $2^{-2j} \leq 2^{-2C} \leq |c_0|^2/100$, this implies that $|e_2^t U_0 p| \geq |c_0|/2$. Finally we note that since both $x, y \in \gamma_z^\epsilon$, the slope of the line joining x and y differs from that of p by

$$(7.13) \quad \lesssim \frac{|\epsilon - z^t(R - Q^t P^{-1} Q)z|}{|x + P^{-1} Qz|^2} \lesssim \frac{\nu_2 2^{\ell-2j-2k}}{(2^{\ell-j-k})^2} = \nu_2 2^{-\ell} \lesssim c' 2^{\ell-j},$$

where we have used (7.8) at the second step. The righthand side is clearly $\leq |c_0|/4$ for c' sufficiently small, therefore,

$$\left| e_2^t U_0 \frac{(x-y)}{|x-y|} \right| \geq \frac{|c_0|}{4}, \text{ i.e., } |u_2 - v_2| \gtrsim n\sigma_1, \text{ since } |x-y| \sim n\sigma_1.$$

This in particular implies (7.9).

Next we assume that $S''_{xz}(\cdot, \cdot)$ vanishes on \mathcal{L}_R^0 . It follows from (7.11) and (7.12) that for $\ell \geq C$,

$$\text{dist}((\alpha_0, \gamma_0), \mathcal{L}_R^0) \gtrsim 2^{\ell-j-k}, \quad \text{dist}((\beta_0, \gamma_0), \mathcal{L}_R^0) \gtrsim 2^{\ell-j-k}.$$

Therefore once again using the linearity of S''_{xz} we obtain

$$|A_0^t p| \gtrsim 2^{\ell-j-k}.$$

Following the same steps as before yields

$$\begin{aligned} 2^{2\ell-2j-2k} &\lesssim |A_0^t p|^2 = |d_1|^2 |e_1^t U_0 p|^2 + |d_2|^2 |e_2^t U_0 p|^2 \\ &\lesssim 2^{-2j-2k} + 2^{-2k} |e_2^t U_0 p|^2, \end{aligned}$$

from which we deduce that $|e_2^t U_0 p| \gtrsim 2^{\ell-j}$. We know in view of (7.13) that

$$\left| \frac{x-y}{|x-y|} - p \right| \lesssim c' 2^{\ell-j},$$

therefore once again by choosing c' sufficiently small we conclude that $|u_2 - v_2| \gtrsim 2^{\ell-j} |x-y| \sim 2^{\ell-j} n\sigma_1$. This completes the proof of the claim (7.9).

In view of the claim, we can estimate $\|\mathcal{T}_{\mathbf{Q}} \mathcal{T}_{\mathbf{Q}'}^*\|$ as follows,

$$\|\mathcal{T}_{\mathbf{Q}} \mathcal{T}_{\mathbf{Q}'}^*\| \leq \sup_y \int \left| K_{\mathcal{T}_{\mathbf{Q}} \mathcal{T}_{\mathbf{Q}'}^*}(x, y) \right| dx \leq \sup_{v \in U_0 \pi_L(\mathbf{Q}')} \mathcal{I},$$

where for $v \in U_0 \pi_L \mathbf{Q}'$,

$$\begin{aligned} \mathcal{I} &= \mathcal{I}(v, n) = \iint_{\mathcal{U}} \frac{C_N \sigma_1 \sigma_0 du}{\prod_{r=1}^2 (1 + \lambda \sigma_1 \sigma_r |u_r - v_r|)^N}, \quad \text{and} \\ \mathcal{U} &= \mathcal{U}(v) = \{u : |u - U_0 c_x(\mathbf{Q})| \leq C \sigma_1, |u_2 - v_2| \gtrsim n \sigma_1\}. \end{aligned}$$

Therefore, it suffices to prove that

$$\begin{aligned} \sum_n \sup_v \sqrt{\mathcal{I}} &\lesssim \sum_{n \lesssim 2^\ell} \left[\frac{\sigma_0 \sigma_1}{(1 + \lambda \sigma_1^2 \sigma_2 n 2^{\ell-j})^N} \min\left(\sigma_1, \frac{1}{\lambda \sigma_1^2}\right) \min\left(\sigma_1, \frac{1}{\lambda \sigma_1 \sigma_2}\right) \right]^{\frac{1}{2}} \\ &\lesssim (\sigma_0 \sigma_1)^{\frac{1}{2}} \min\left(\sigma_1, \frac{1}{\lambda \sigma_1^2}\right)^{\frac{1}{2}} \min\left(\sigma_1, \frac{1}{\lambda \sigma_1 \sigma_2}\right)^{\frac{1}{2}} \min\left(\frac{1}{\lambda \sigma_1^2 \sigma_2 2^{\ell-j}}, 2^\ell\right) \end{aligned}$$

is summable in ℓ, j and k , with the desired sum of $C \lambda^{-\frac{2}{3}}$. The following cases arise.

Case 1 : $\lambda \sigma_1^3 \geq 1$, i.e., $2^{3j} \leq \lambda 2^{-3k}$. (This in particular implies that $\lambda 2^{-3k} \geq 1$). In this case,

$$\begin{aligned} \sum_n \sup_v \sqrt{\mathcal{I}} &\lesssim (\sigma_0 \sigma_1)^{\frac{1}{2}} \left(\frac{1}{\lambda \sigma_1^2} \cdot \frac{1}{\lambda \sigma_1 \sigma_2} \right)^{\frac{1}{2}} \min\left(\frac{1}{\lambda \sigma_1^2 \sigma_2 2^{\ell-j}}, 2^\ell\right) \\ &\lesssim \begin{cases} \lambda^{-1} \sigma_1^{-1} \sigma_0^{\frac{1}{2}} \sigma_2^{-\frac{1}{2}} 2^\ell & \text{if } 2^\ell \leq (\lambda \sigma_1^2 \sigma_2 2^{\ell-j})^{-1}, \\ \lambda^{-1} \sigma_1^{-1} \sigma_0^{\frac{1}{2}} \sigma_2^{-\frac{1}{2}} \frac{1}{\lambda \sigma_1^2 \sigma_2 2^{\ell-j}} & \text{if } 2^\ell > \lambda \sigma_1^2 \sigma_2 2^{\ell-j}, \end{cases} \\ &\lesssim \begin{cases} \lambda^{-1} 2^{\frac{3\ell}{2} + k} & \text{if } 2^\ell \leq \lambda^{-\frac{1}{2}} 2^{\frac{3j+3k}{2}} \\ \lambda^{-2} 2^{-\frac{\ell}{2} + 3j + 4k} & \text{if } 2^\ell > \lambda^{-\frac{1}{2}} 2^{\frac{3j+3k}{2}} \end{cases}. \end{aligned}$$

Summing in ℓ , we get

$$\left\{ \begin{array}{l} \lambda^{-1} (\lambda^{-\frac{1}{2}} 2^{\frac{3j+3k}{2}})^{\frac{3}{2}} 2^k \\ \lambda^{-2} (\lambda^{-\frac{1}{2}} 2^{\frac{3j+3k}{2}})^{-\frac{1}{2}} 2^{3j+4k} \end{array} \right\} = \lambda^{-\frac{7}{4}} 2^{\frac{9j}{4} + \frac{13k}{4}}$$

in both cases. Summing in j and k now yields

$$\sum_{\lambda 2^{-3k} \geq 1} \sum_{2^{3j} \leq \lambda 2^{-3k}} \lambda^{-\frac{7}{4}} 2^{\frac{9j}{4} + \frac{13k}{4}} \lesssim \sum_{\lambda 2^{-3k} \geq 1} \lambda^{-\frac{7}{4}} (\lambda 2^{-3k})^{\frac{3}{4}} 2^{\frac{13k}{4}} = \sum_{\lambda 2^{-3k} \geq 1} \lambda^{-1} 2^k \lesssim \lambda^{-\frac{2}{3}}.$$

Case 2 : $\lambda \sigma_1^3 < 1$ but $\lambda \sigma_1^2 \sigma_2 \geq 1$, i.e., $\lambda 2^{-3j-3k} < 1$, $\lambda 2^{-2j-3k} \geq 1$. In this case,

$$\begin{aligned} \sum_n \sqrt{\sup_v \mathcal{I}} &\lesssim (\sigma_0 \sigma_1)^{\frac{1}{2}} \sigma_1^{\frac{1}{2}} \left(\frac{1}{\lambda \sigma_1 \sigma_2} \right)^{\frac{1}{2}} \min\left(\frac{1}{\lambda \sigma_1^2 \sigma_2 2^{\ell-j}}, 2^\ell\right) \\ &\lesssim \begin{cases} \lambda^{-\frac{1}{2}} 2^{\frac{3\ell}{2} - \frac{3j}{2} - \frac{k}{2}} & \text{if } 2^\ell \leq \lambda^{-\frac{1}{2}} 2^{\frac{3j+3k}{2}} \\ \lambda^{-\frac{3}{2}} 2^{-\frac{\ell}{2} + \frac{3j}{2} + \frac{5k}{2}} & \text{if } 2^\ell > \lambda^{-\frac{1}{2}} 2^{\frac{3j+3k}{2}} \end{cases}. \end{aligned}$$

In both cases, the sum in ℓ gives

$$\left\{ \begin{array}{l} \lambda^{-\frac{1}{2}} \lambda^{-\frac{3}{4}} 2^{\frac{9j+9k}{4}} 2^{-\frac{3j}{2} - \frac{k}{2}} \\ \lambda^{-\frac{3}{2}} \lambda^{\frac{1}{4}} 2^{-\frac{3j+3k}{4}} 2^{\frac{3j}{2} + \frac{5k}{2}} \end{array} \right\} = \lambda^{-\frac{5}{4}} 2^{\frac{3j}{4} + \frac{7k}{4}}.$$

Now summing in j and k we obtain

$$\sum_{\lambda 2^{-3k} \geq 1} \sum_{2^{2j} \leq \lambda 2^{-3k}} \lambda^{-\frac{5}{4}} 2^{\frac{3j}{4} + \frac{7k}{4}} \lesssim \sum_{\lambda 2^{-3k} \geq 1} \lambda^{-\frac{5}{4}} (\lambda 2^{-3k})^{\frac{3}{8}} 2^{\frac{7k}{4}} = \lambda^{-\frac{7}{8}} \sum_{\lambda 2^{-3k} \geq 1} 2^{\frac{5k}{8}} \lesssim \lambda^{-\frac{2}{3}}.$$

Case 3 : $\lambda \sigma_1^2 \sigma_2 < 1$, i.e., $\lambda 2^{-2j-3k} < 1$. In this case,

$$\begin{aligned} \sum_n \sqrt{\sup_v I} &\lesssim (\sigma_0 \sigma_1)^{\frac{1}{2}} \sigma_1^{\frac{1}{2}} \sigma_1^{\frac{1}{2}} \min \left(\frac{1}{\lambda \sigma_1^2 \sigma_2 2^{\ell-j}}, 2^\ell \right) \\ &\lesssim \begin{cases} 2^{\frac{3\ell}{2} - \frac{5j}{2} - 2k} & \text{if } 2^\ell \leq \min \left(\lambda^{-\frac{1}{2}} 2^{\frac{3j+3k}{2}}, 2^j \right) \\ \lambda^{-1} 2^{-\frac{\ell}{2} + \frac{j}{2} + k} & \text{if } 2^\ell > \lambda^{-\frac{1}{2}} 2^{\frac{3j+3k}{2}} \end{cases}. \end{aligned}$$

Subcase 1 : Suppose $2^\ell \leq \min(\lambda^{-1/2} 2^{(3j+3k)/2}, 2^j)$. If $\lambda^{-1/2} 2^{(3j+3k)/2} \leq 2^j$, then $2^j \leq \lambda 2^{-3k}$, which in particular implies that $\lambda 2^{-3k} \geq 1$. Therefore,

$$\begin{aligned} \sum_{k,j} \sum_{2^\ell \leq \lambda^{-\frac{1}{2}} 2^{\frac{3j+3k}{2}}} 2^{\frac{3\ell}{2} - \frac{5j}{2} - 2k} &\lesssim \sum_{k,j} \lambda^{-\frac{3}{4}} 2^{\frac{9j+9k}{4}} 2^{-\frac{5j}{2} - 2k} \\ &\lesssim \sum_k \sum_{2^{2j} > \lambda 2^{-3k}} \lambda^{-\frac{3}{4}} 2^{-\frac{j}{4} + \frac{k}{4}} \\ &\lesssim \sum_{\lambda 2^{-3k} \geq 1} \lambda^{-\frac{3}{4}} (\lambda^{-1} 2^{3k})^{\frac{1}{8}} 2^{\frac{k}{4}} \\ &\lesssim \lambda^{-\frac{3}{4} - \frac{1}{8}} \sum_{\lambda 2^{-3k} \geq 1} 2^{\frac{5k}{8}} \lesssim \lambda^{-\frac{2}{3}}. \end{aligned}$$

If $2^j \leq \lambda^{-1/2} 2^{(3j+3k)/2}$ then $2^j \geq \lambda 2^{-3k}$. The summation here proceeds as follows,

$$\begin{aligned} \sum_{k,j} \sum_{\ell \leq j} 2^{\frac{3\ell}{2} - \frac{5j}{2} - 2k} &\lesssim \sum_k \sum_{2^j \geq \lambda 2^{-3k}} 2^{-j-2k} \\ &\lesssim \sum_k \begin{cases} \lambda^{-1} 2^{3k} 2^{-2k} & \text{if } \lambda 2^{-3k} \geq 1 \\ 2^{-2k} & \text{if } \lambda 2^{-3k} < 1 \end{cases} \\ &\lesssim \lambda^{-\frac{2}{3}}. \end{aligned}$$

Subcase 2 : Suppose $2^\ell > \lambda^{-1/2} 2^{(3j+3k)/2}$. Then

$$\sum_\ell \lambda^{-1} 2^{-\frac{\ell}{2} + \frac{j}{2} + k} \lesssim \lambda^{-1} \lambda^{\frac{1}{4}} 2^{-\frac{3j+3k}{4}} 2^{\frac{j}{2} + k} = \lambda^{-\frac{3}{4}} 2^{-\frac{j}{4} + \frac{k}{4}}.$$

We now follow the same steps as in the first part of Subcase 1 to obtain the desired bound of $\lambda^{-\frac{2}{3}}$. □

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