The Simplex Method in Complete Detail

Setup: Start with a dictionary whose corresponding basic solution is *feasible*, i.e., nonnegative. Name the corresponding basis $\mathcal{B}^0$. The dictionary is

$$\begin{array}{ll} f = v^0 + \displaystyle\sum_{j\in\mathcal{N}^0} c_j^0 x_j & \\ \hline x_i = b_i^0 - \displaystyle\sum_{j\in\mathcal{N}^0} a_{ij}^0 x_j & (i \in \mathcal{B}^0) \end{array} \tag{1}$$

The corresponding basic feasible solution (BFS) is $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_n^0)$, where

$$x_i^0 = b_i^0 \quad \forall i \in \mathcal{B}^0, \qquad x_j^0 = 0, \quad \forall j \in \mathcal{N}^0.$$

Its value is $f(\mathbf{x}^0) = v^0$. Feasibility guarantees $b_i^0 \ge 0$ for each $i \in \mathcal{B}^0$.

Every vector $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}$ can be generated by assigning arbitrary real values to the the nonbasic variables on the right in this dictionary, and then "looking up" the resulting values for the basic ones by plugging the chosen numbers into the equations. Generating *feasible* solutions $\mathbf{x}$, where $\mathbf{x} \ge \mathbf{0}$ componentwise, requires more careful choices for the nonbasic values. These considerations are central to the Simplex Method.

1. **Test for Optimality:** If $c_j^0 \le 0$ for each $j \in \mathcal{N}^0$, then $f(\mathbf{x}) \le v^0$ for every feasible solution $\mathbf{x}$. Therefore $f(\mathbf{x}^0) = f_{\max} = v^0$, and the current BFS $\mathbf{x}^0$ is a maximizer. In this case we can say a little more:

   - ☐ If $c_j^0 < 0$ for each $j \in \mathcal{N}^0$, then a positive choice for any x_j, $j \in \mathcal{N}^0$, will reduce the objective value. Therefore the maximum value, v^0, can be realized only by choosing $x_j = 0$ for each $j \in \mathcal{N}^0$. The maximizing BFS is unique; the problem is solved. Stop and report.

   - ☐ If $c_j^0 = 0$ for some $j \in \mathcal{N}$, then the corresponding nonbasic variable x_j has no influence on the objective value. Any choice of $x_j > 0$ compatible with all the other constraints will produce a maximizing point (though not necessarily a basic one). Hence there may be other inputs, not necessarily basic, that share with $\mathbf{x}^0$ the honour of providing the maximum value for f. The question of how aggressively to pursue the job of identifying *all* the maximizing inputs depends on the problem at hand and the inclination of the solver. Stop and report.

2. **Choose an Entering Index.** Scan the objective row in the dictionary looking at the coefficients of the nonbasic variables. At least one must be positive, or we would have stopped in Step 1. Now each index $j \in \mathcal{N}$ that has $c_j^0 > 0$ is eligible to enter the basis. Pick one such j, and use the special letter $E = j$ to identify it. ("E" stands for "Entering".) For now, any choice is acceptable, but it's nice to have a systematic and predictable approach. Reasonable policies include the following:

   (a) [The Largest-Coefficient Rule – "Anstee's"] Always choose the eligible variable for which c_j^0 is biggest. Break ties by choosing the smallest subscript.

(b) [The smallest-subscript rule – "Bland's"] Always choose the eligible variable for which j is smallest (even if its coefficient c_j^0 is not the largest).

(c) [The Lazy Rule] Choose the eligible variable giving the simplest pivot equation.

3. Choose a Leaving Index. Focus on the entering variable x_E, which is currently nonbasic, by rewriting the dictionary above as follows:

$$\begin{array}{l} f = v^0 + \; c_E^0 x_E + \displaystyle\sum_{j\in\mathcal{N}^0\setminus\{E\}} c_j^0 x_j \\ \hline x_i = b_i^0 - a_{iE}^0 x_E - \displaystyle\sum_{j\in\mathcal{N}^0\setminus\{E\}} a_{ij}^0 x_j \qquad (i\in\mathbb{B}^0) \end{array}$$

Now imagine setting $x_E = t$, but keeping all other nonbasic variables at 0. Feasibility restricts our choices to $t \geq 0$. Then the dictionary shows how all the variables must change as t varies:

$$\begin{aligned} &x_E(t) = t; \quad x_j(t) = 0, && j \in \mathcal{N}^0 \setminus \{E\}\,; \\ &x_i(t) = b_i^0 - t a_{iE}^0, && i \in \mathcal{B}^0. \end{aligned} \tag{2}$$

Meanwhile, the objective value depends on t as follows:

$$f(\mathbf{x}(t)) = v^0 + c_E^0 t. \tag{3}$$

Restrictions on which t-values are allowed here will come from feasibility requirements for the basic variables.

For each basic index $i \in \mathcal{B}^0$ where $a_{iE}^0 \leq 0$, we will have $x_i(t) \geq b_i^0 \geq 0$ for all $t \geq 0$. Such indices contribute no feasibility restrictions.

- □ If *every* index $i \in \mathcal{B}^0$ has $a_{iE}^0 \leq 0$, then there are no feasibility restrictions at all, so t can be allowed to take any nonnegative value. Since $c_E^0 > 0$, line (3) shows that there is no limit to how large $f(\mathbf{x}(t))$ can be made. The problem is called "unbounded". Report this fact, perhaps with some supporting analysis, and stop.
- □ Suppose the inequality $a_{iE}^0 > 0$ holds for *one or more* indices $i \in \mathcal{B}^0$. For every such index i, the basic variable $x_i(t)$ will decrease as t grows, with

$$x_i(t) = 0 \iff t = t_i \stackrel{\text{def}}{=} b_i^0 / a_{iE}^0 \geq 0.$$

  To keep all the choices in (2) feasible we must insist that

$$0 \leq t \leq \min_i \left\{ t_i \,:\, i \in \mathcal{B}^0,\ a_{iE} > 0 \right\}. \tag{4}$$

  To get maximum payoff from the entering index E, we choose the biggest value of t allowed here, namely,

$$t = \min_i \left\{ \frac{b_i^0}{a_{iE}^0} \,:\, i \in \mathcal{B}^0,\ a_{iE}^0 > 0 \right\}. \tag{5}$$

  All the indices i at which this minimum is achieved are candidates for leaving variables. Any one will work, but it's nice to have a systematic choice. So let's break ties by picking the smallest such subscript. Define $L = i$ for this smallest subscript. ("L" stands for "Leaving".) This makes the choice of t in line (5) explicit:

$$t = \frac{b_L^0}{a_{LE}^0}. \tag{6}$$

Degeneracy: It can happen that $\min\{t_i\} = 0$, because $b_i^0 = 0$ for some relevant i. In such cases, Choose $L = i$ anyway and proceed; more discussion later.

4. **Update BFS and Dictionary:** Use the symbol $\mathcal{B}^+$ for the new basis built from $\mathcal{B}^0$ by pivoting L out and E in:

$$\mathcal{B}^+ = \left(\mathcal{B}^0 \setminus \{L\}\right) \cup \{E\}. \tag{7}$$

The new BFS will be the vector $\mathbf{x}^+$, with nonbasic components

$$x_L^+ = 0, \quad x_j^+ = 0 \text{ for } j \in \mathcal{N}^0 \setminus \{E\}. \tag{8}$$

The basic components of $\mathbf{x}^+$ are determined by the choice of t in line (6):

$$\begin{aligned} x_E^+ &= \frac{b_L^0}{a_{LE}^0} \qquad \text{(rearrange the line where } i = L\text{)} \\ x_i^+ &= b_i^0 - \left(\frac{b_L^0}{a_{LE}^0}\right) a_{iE}^0, \text{ for } i \in \mathcal{B}^0 \setminus \{L\}; \end{aligned} \tag{9}$$

Finally, the new BFS will have an objective value of

$$f(\mathbf{x}^+) = v^0 + c_E^0 \left(\frac{b_L^0}{a_{LE}^0}\right). \tag{10}$$

Monotonicity: Recall that $c_E^0 > 0$ and $a_{LE}^0 > 0$ and $b_L^0 \geq 0$, so $f(\mathbf{x}^+) \geq f(\mathbf{x}^0)$. The new objective value is definitely not smaller than the original one. Indeed, whenever $b_L^0 > 0$, the new objective value is strictly higher than the original. So when $b_L^0 > 0$ the basis $\mathcal{B}^0$ is left behind forever: its unique objective value will never be seen again. (This is an important ingredient in proving that the simplex algorithm eventually stops.)

Individual indices can bounce in and out of the basis several times as the algorithm proceeds, but the basis *as a set* will never re-appear.

Degeneracy: It may happen that $b_L^0 = 0$, so that the t-value in line (6) is $t = 0$, and lines (8)–(9) show that the new value of x_L (which has left the basis and now gets the value 0) is the same as its previous value, which was b_L^0. The new value of x_E (which is entering the basis but started with the value 0) is also the same as its previous value. The same goes for all other indices. So the new BFS $\mathbf{x}^+$ is identical to the original one, $\mathbf{x}^0$, and of course it follows that the new objective value, $f(\mathbf{x}^+)$, is also identical to the original one. Although these key elements are unchanged, there is an important difference: line (7) shows that the classification of indices as "basic" or "nonbasic" has been modified. In many problems, this sets up a favourable starting point for nondegenerate steps in the future. (But there is more to be said on this point.)