The Simplex Method in Matrix Notation

This is also known as "the Revised Simplex Method". Matrix Notation gives ...

1. Conceptual clarity on stuff we know;
2. Computational accuracy and efficiency; and
3. Streamlined access to some new material (like sensitivity analysis).

Setup—Standard Equality-Style Problem. A matrix $A \in \mathbb{R}^{m\times n}$ is given, along with vectors $\mathbf{b} \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

$$(P) \quad \max\left\{\mathbf{c}^T\mathbf{x} \,:\, A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0\right\}.$$

Always assume A has linearly independent rows. This requires $m \leq n$.

Geometric Interpretation. Label the n columns of matrix A:

$$A = \left[\mathbf{a}^{(1)} \mid \mathbf{a}^{(2)} \mid \cdots \mid \mathbf{a}^{(n)}\right].$$

Given some $\mathbf{x} \in \mathbb{R}^n$, recognize

$$A\mathbf{x} = \left[\mathbf{a}^{(1)} \mid \mathbf{a}^{(2)} \mid \cdots \mid \mathbf{a}^{(n)}\right]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}^{(1)} + x_2\mathbf{a}^{(2)} + \cdots + x_n\mathbf{a}^{(n)}.$$

That's a *linear combination of cols from A.* So, "Solve for x in $A\mathbf{x} = \mathbf{b}$" means, "Express the given vector b as a linear combination of the cols in A." Each col is in $\mathbb{R}^m$, and we have n cols to choose between (with $n \geq m$). For a *basic solution,* pick any m lin indep cols, call them the basis (truly a basis for $\mathbb{R}^m$, the space containing the given rhs $\mathbf{b}$), and use only those. That's a "basic solution (BS) for $A\mathbf{x} = \mathbf{b}$"; there may be several of these.

Each component c_j in vector $c \in \mathbb{R}^n$ tells the reward (if $c_j > 0$) or penalty (if $c_j < 0$) for including column $\mathbf{a}^{(j)}$ in the representation.

Simplex Dictionary. Pick any m linearly independent columns from A. Record their subscripts in a set $\mathcal{B}$. Call these the basis. Permute/clump selected cols into square matrix B (size $m \times m$). Symbolic decomposition

$$A \stackrel{\text{def}}{=} [A_{\mathcal{B}} \mid A_{\mathcal{N}}] \stackrel{\text{def}}{=} [B \mid N], \qquad \mathbf{x} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{x}_{\mathcal{B}} \\ \hline \mathbf{x}_{\mathcal{N}} \end{bmatrix}, \qquad c \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{c}_{\mathcal{B}} \\ \hline \mathbf{c}_{\mathcal{N}} \end{bmatrix},$$

leads to

$$A\mathbf{x} = \mathbf{b} \iff B\mathbf{x}_{\mathcal{B}} = \mathbf{b} - N\mathbf{x}_{\mathcal{N}} \iff \mathbf{x}_{\mathcal{B}} = B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_{\mathcal{N}}.$$

Track the objective:

$$\begin{aligned} f = \mathbf{c}^T\mathbf{x} = \mathbf{c}_{\mathcal{B}}^T\mathbf{x}_{\mathcal{B}} + \mathbf{c}_{\mathcal{N}}^T\mathbf{x}_{\mathcal{N}} &= \mathbf{c}_{\mathcal{B}}^T\left(B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_{\mathcal{N}}\right) + \mathbf{c}_{\mathcal{N}}^T\mathbf{x}_{\mathcal{N}} \\ &= \mathbf{c}_{\mathcal{B}}^TB^{-1}\mathbf{b} - \left(\mathbf{c}_{\mathcal{B}}^TB^{-1}N - \mathbf{c}_{\mathcal{N}}^T\right)\mathbf{x}_{\mathcal{N}}. \end{aligned}$$

New dictionary

$$\begin{array}{rll} f = & \mathbf{c}_{\mathcal{B}}^TB^{-1}\mathbf{b} - & \left(\mathbf{c}_{\mathcal{B}}^TB^{-1}N - \mathbf{c}_{\mathcal{N}}^T\right)\mathbf{x}_{\mathcal{N}} \\ \hline \mathbf{x}_{\mathcal{B}} = & B^{-1}\mathbf{b} - & B^{-1}N\mathbf{x}_{\mathcal{N}}. \end{array}$$

... an algebraic structure containing symbols $x_1, \ldots, x_n, f$ together with a bunch of numbers. Substituting $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$ determines the numerical values of $\mathbf{x}_{\mathcal{B}}^* = B^{-1}\mathbf{b}$ in a Basic Solution for $A\mathbf{x} = \mathbf{b}$. It's nice when this is feasible, but the derivation above does not require that.

Versatility. Imagine picking any values at all for the $n-m$ elements of vector $\mathbf{x}_{\mathcal{N}}$. This dictionary shows the m components of $\mathbf{x}_{\mathcal{B}}$ you must use to satisfy $A\mathbf{x}=\mathbf{b}$ and the value of $f=\mathbf{c}^T\mathbf{x}$ that results when you do. (Feasibility is ignored.) Choosing $\mathbf{x}^*_{\mathcal{N}}=\mathbf{0}$ gives the basic solution $\mathbf{x}^*_{\mathcal{B}}=B^{-1}\mathbf{b}$ and the objective value $f^*=\mathbf{c}^T\mathbf{x}^*=\mathbf{c}^T_{\mathcal{B}}\mathbf{x}^*_{\mathcal{B}}=\mathbf{c}^T_{\mathcal{B}}B^{-1}\mathbf{b}$.

Optimality and Improvability. Suppose $\mathbf{x}^*_{\mathcal{B}}\geq 0$, so $\mathbf{x}^*$ is feasible. Consider the row vector $\mathbf{z}^T_{\mathcal{N}}=(\mathbf{c}^T_{\mathcal{B}}B^{-1}N-\mathbf{c}^T_{\mathcal{N}})$:

- If each component $z_i>0$, every nonzero vector $x_{\mathcal{N}}$ will subtract a positive amount from f^*. So $\mathbf{x}^*$ is the unique maximizer.
- If each component $z_i\geq 0$, no nonzero vector $\mathbf{x}_{\mathcal{N}}$ can add a positive amount to f^*. So x^* is a maximizer, but perhaps not unique.
- If some component $z_k<0$, a positive value of component $(x_{\mathcal{N}})_k$ will make f increase. Improvement is possible, so we will pivot.

Pivot Selection. Suppose we have a feasible dictionary and the cost coefficient row $\mathbf{z}^T_{\mathcal{N}}=(\mathbf{c}^T_{\mathcal{B}}B^{-1}N-\mathbf{c}^T_{\mathcal{N}})$ has a negative entry. Identify the associated column of A and call it $\mathbf{a}^{(E)}$. Notice $E\in\mathcal{N}$. Choosing $\mathbf{x}_{\mathcal{N}}(t)$ so that $x_E=t$ and $x_j=0$ for all other $j\in\mathcal{N}$ leads to $N\mathbf{x}_{\mathcal{N}}(t)=t\mathbf{a}^{(E)}$. Track the values of current basic coeffs:

$$\mathbf{x}_{\mathcal{B}}(t)=B^{-1}\mathbf{b}-B^{-1}[t\mathbf{a}^{(E)}]. \qquad (**)$$

For many t this will have m nonzero entries. Pick the smallest $t\geq 0$ where one or more entry changes to 0: those entries identify the columns eligible to leave. Call the lucky value t^*.

Update. Swap entering col into basis; your chosen leaving col out; update coefficient vector using $\mathbf{x}^*_E=t^*$, other coeffs from $(**)$.

Efficient Implementation. One never actually computes B^{-1} from B. (Sometimes B^{-1} is given on exams.) Instead, we solve linear systems as follows:

(i) To generate objective coefficients, introduce y to reorganize

$$\mathbf{c}^T_{\mathcal{B}}B^{-1}N=\mathbf{y}^TN, \quad \text{as} \quad \mathbf{y}^T=\mathbf{c}^T_{\mathcal{B}}B^{-1}, \quad \text{i.e.,} \quad \mathbf{y}^TB=\mathbf{c}^T_{\mathcal{B}}.$$

Find $\mathbf{y}$ by solving a linear system.

(ii) In coefficient update, find $\mathbf{d}=B^{-1}\mathbf{a}^{(E)}$ by solving system

$$B\mathbf{d}=\mathbf{a}^{(E)}.$$

Use tiny labels called "basis headers" to remember which cols are basic in each iteration. [Chvátal box 7.1, page 103.]

The Revised Simplex Method, Step by Step

Context. The Revised Simplex Method works on problems of this form:

$$\text{(EqLP)} \qquad \max\left\{\mathbf{c}^T\mathbf{x} \,:\, A\mathbf{x}=\mathbf{b},\ \mathbf{x}\geq\mathbf{0}\right\}.$$

(Many problems can be put into this form.) Here a matrix A of shape $m\times n$ is given, along with (column) vectors $\mathbf{c}\in\mathbb{R}^n$, $\mathbf{b}\in\mathbb{R}^m$. We assume that A has linearly independent rows (so $m\leq n$).

Initialize. You need a feasible basis to get started. (Try to guess one; use an auxiliary "Phase One" step if guessing fails.) Let $\mathcal{B}$ be the set of m subscripts that define the current basis; let $\mathcal{N}$ be the set containing the $n-m$ non-basic subscripts.

Partition. Use the sets of subscripts $\mathcal{B}$ and $\mathcal{N}$ to select columns from A and their corresponding rows of $\mathbf{x}$, $\mathbf{c}$:

$$A=\left[\,A_{\mathcal{B}}\quad A_{\mathcal{N}}\,\right]=\left[\,B\quad N\,\right],\qquad \mathbf{c}=\begin{bmatrix}\mathbf{c}_{\mathcal{B}}\\ \mathbf{c}_{\mathcal{N}}\end{bmatrix},\qquad \mathbf{x}=\begin{bmatrix}\mathbf{x}_{\mathcal{B}}\\ \mathbf{x}_{\mathcal{N}}\end{bmatrix}.$$

Matrix B has shape $m\times m$. It is certain to be invertible because the columns of A with indices in $\mathcal{B}$ must be linearly independent. The dictionary is

$$\begin{array}{rcl} f = & \mathbf{c}_{\mathcal{B}}^TB^{-1}\mathbf{b} - & \left(\mathbf{c}_{\mathcal{B}}^TB^{-1}N-\mathbf{c}_{\mathcal{N}}^T\right)\mathbf{x}_{\mathcal{N}} \\ \hline \mathbf{x}_{\mathcal{B}} = & B^{-1}\mathbf{b} - & B^{-1}N\mathbf{x}_{\mathcal{N}} \end{array}$$

The current BFS $\mathbf{x}^*\in\mathbb{R}^n$ has blocks $\mathbf{x}^*_{\mathcal{B}}=B^{-1}\mathbf{b}\in\mathbb{R}^m$, $\mathbf{x}^*_{\mathcal{N}}=\mathbf{0}\in\mathbb{R}^{n-m}$.

Select Entering Col. Scan $\mathbf{z}_{\mathcal{N}}^T=\mathbf{c}_{\mathcal{B}}^TB^{-1}N-\mathbf{c}_{\mathcal{N}}^T$ for negative entries. To find $\mathbf{z}_{\mathcal{N}}$,

(i) Find $\mathbf{y}^T=\mathbf{c}_{\mathcal{B}}^TB^{-1}$ (a row vector) by solving the system $\mathbf{y}^TB=\mathbf{c}_{\mathcal{B}}^T$.

(ii) Build $\mathbf{z}_{\mathcal{N}}^T=\mathbf{y}^TN-\mathbf{c}_{\mathcal{N}}^T$.

If no entry is negative, current BFS is optimal. Stop. If negative entries exist, pick one. It labels a suitable "entering column" which has an original index E. The column itself is $\mathbf{a}^{(E)}$, with coefficient x_E.

Select Leaving Col. Set nonbasic coefficient $x_E=t$, while keeping all other nonbasic vars at 0. This gives the vector $\mathbf{x}_{\mathcal{N}}=\mathbf{x}_{\mathcal{N}}(t)$ exactly one nonzero entry, with

$$N\mathbf{x}_{\mathcal{N}}(t)=t\mathbf{a}^{(E)}.$$

Watch the coefficients of the current basis change as $t\geq 0$ increases:

$$\mathbf{x}_{\mathcal{B}}(t)=\mathbf{x}_{\mathcal{B}}(0)-B^{-1}N\mathbf{x}_{\mathcal{N}}(t)=\mathbf{x}^*_{\mathcal{B}}-tB^{-1}\mathbf{a}^{(E)}=\mathbf{x}^*_{\mathcal{B}}-t\mathbf{d}, \tag{$**$}$$

where $\mathbf{d}$ is found by solving $B\mathbf{d}=\mathbf{a}^{(E)}$. Let t^* denote the smallest $t\geq 0$ for which $\mathbf{x}_{\mathcal{B}}(t)$ has a zero entry. Each zero entry in $\mathbf{x}_{\mathcal{B}}(t^*)$ identifies a column in the current basis that may leave. Pick one; use L as a symbol for the leaving index. Then the leaving column is $\mathbf{a}^{(L)}$. (If $\mathbf{x}_{\mathcal{B}}(t)$ never develops a zero entry, sending $t\to\infty$ proves problem is unbounded. Report that fact and stop.)

Update. Indices E and L are column numbers. At the beginning, $E\in\mathcal{N}$ and $L\in\mathcal{B}$. Swap these two. Then go back to original problem and make the new block matrices $B=A_{\mathcal{B}}$, $N=A_{\mathcal{N}}$, and cost vectors $\mathbf{c}_{\mathcal{B}}^T$, $\mathbf{c}_{\mathcal{N}}^T$. Update the BFS $\mathbf{x}^*$ by noting $\mathbf{x}^*_{\mathcal{N}}=\mathbf{0}$; $m-1$ entries for $\mathbf{x}_{\mathcal{B}}(t^*)$ appear in $(**)$; and $x_E=t^*$. Loop back to "Select Entering Col".

Example. Here are some RSM pivots using Bland's rule for the problem

$$
\begin{aligned}
&\text{maximize } f = 3x_1 + 2x_2 + 4x_3\\
&\text{subject to} \quad
\begin{array}{rrrrrrl}
x_1 + & x_2 + & 2x_3 + & x_4 & & & = 4\\
2x_1 & & + 3x_3 & & + x_5 & & = 5\\
2x_1 + & x_2 + & 3x_3 & & & + x_6 & = 7\\
x_j \geq 0 & & & & & &
\end{array}
\end{aligned}
$$

Solution. Setup:

$$
A = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0\\ 2 & 0 & 3 & 0 & 1 & 0\\ 2 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4\\ 5\\ 7 \end{bmatrix},
$$
$$
\mathbf{c}^T = [3 \quad 2 \quad 4 \quad 0 \quad 0 \quad 0].
$$

An easy BFS comes from choosing $\mathcal{B} = \{4, 5, 6\}$, so $\mathcal{N} = \{1, 2, 3\}$, giving

$$
B = \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix} \end{array}, \qquad N = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 2\\ 2 & 0 & 3\\ 2 & 1 & 3 \end{pmatrix} \end{array},
$$

$$
\mathbf{c}_{\mathcal{B}}^T = \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \end{array}, \qquad \mathbf{c}_{\mathcal{N}}^T = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 3 & 2 & 4 \end{pmatrix} \end{array},
$$

$$
\mathbf{x}_{\mathcal{B}}^* = B^{-1}\mathbf{b} = \begin{array}{c} x_4\\ x_5\\ x_6 \end{array} \begin{pmatrix} 4\\ 5\\ 7 \end{pmatrix}, \qquad f^* = 0.
$$

First iteration:

(1) Select entering variable using objective coeffs $\mathbf{c}_{\mathcal{N}}^T - \mathbf{c}_{\mathcal{B}}^T B^{-1} N$.

(a) Solve for $\mathbf{y}$ in

$$
\mathbf{y}^T = \mathbf{c}_{\mathcal{B}}^T B^{-1}, \quad \text{i.e.,} \quad \mathbf{y}^T B = \mathbf{c}_{\mathcal{B}}^T.
$$

Easy: $\mathbf{y}^T = \mathbf{c}_{\mathcal{B}}^T = [0 \quad 0 \quad 0]$.

(b) Write and scan vector

$$
\mathbf{z}_{\mathcal{N}}^T = \mathbf{y}^T N - \mathbf{c}_{\mathcal{N}}^T = -\mathbf{c}_{\mathcal{N}} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} -3 & -2 & -4 \end{pmatrix} \end{array}.
$$

Select x_1 to enter (Bland's Rule). So $E = 1$, $\mathbf{a}^{(E)} = \mathbf{a}^{(1)}$, $x_E = x_1$.

(2) Select leaving variable, noting $N x_{\mathcal{N}} = t\mathbf{a}^{(\text{in})} = t\mathbf{a}^{(1)}$ so $\mathbf{x}_{\mathcal{B}}(t) = \mathbf{x}_{\mathcal{B}}(0) - td$.

(a) Solve for $\mathbf{d}$ in

$$
\mathbf{d} = B^{-1}\mathbf{a}^{(1)}, \quad \text{i.e.,} \quad B\mathbf{d} = \mathbf{a}^{(1)}.
$$

Easy: $\mathbf{d} = \mathbf{a}^{(1)}$.

(b) Monitor

$$\mathbf{x}_{\mathcal{B}}(t) = \begin{array}{c} x_4 \\ x_5 \\ x_6 \end{array} \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} - t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{array}{c} x_4 \\ x_5 \\ x_6 \end{array} \begin{pmatrix} 4-t \\ 5-2t \\ 7-2t \end{pmatrix}.$$

As t increases, $t^* = 5/2$ is the smallest value where $\mathbf{x}_{\mathcal{B}}(t)$ picks up a zero component. (Smallest choice makes sure $\mathbf{x}_{\mathcal{B}}(t^*)$ stays feasible.) Variable x_5 labels that slot, so it will leave. Write $L = 5$, so $\mathbf{x}_L = x_5$, $\mathbf{a}^{(L)} = \mathbf{a}^{(5)}$.

(3) Update everything.

(a) New BFS refers to sets $\mathcal{B}$ and $\mathcal{N}$ above:

$$\mathbf{x}^*(t^*) = \begin{bmatrix} x_{\mathcal{B}}(t^*) \\ x_{\mathcal{N}}(t^*) \end{bmatrix} = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{pmatrix} 5/2 \\ 0 \\ 0 \\ 3/2 \\ 0 \\ 2 \end{pmatrix}.$$

(b) New basis selection sets $\mathcal{B} = \{1,4,6\}$ and $\mathcal{N} = \{2,3,5\}$.

(c) New partitioned matrices

$$B = \begin{array}{c} \begin{array}{ccc} x_1 & x_4 & x_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \end{array}, \qquad N = \begin{array}{c} \begin{array}{ccc} x_2 & x_3 & x_5 \end{array} \\ \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 3 & 0 \end{pmatrix} \end{array},$$

$$\mathbf{c}_{\mathcal{B}}^T = \begin{array}{c} \begin{array}{ccc} x_1 & x_4 & x_6 \end{array} \\ \begin{pmatrix} 3 & 0 & 0 \end{pmatrix} \end{array}, \qquad \mathbf{c}_{\mathcal{N}}^T = \begin{array}{c} \begin{array}{ccc} x_2 & x_3 & x_5 \end{array} \\ \begin{pmatrix} 2 & 4 & 0 \end{pmatrix} \end{array},$$

$$\mathbf{x}_{\mathcal{B}}^* = B^{-1}\mathbf{b} = \begin{array}{c} x_1 \\ x_4 \\ x_6 \end{array} \begin{pmatrix} 5/2 \\ 3/2 \\ 2 \end{pmatrix}, \qquad f^* = 15/2.$$

Second iteration:

(1) Select entering variable.

(a) Solve for y in

$$\mathbf{y}^T = \mathbf{c}_{\mathcal{B}}^T B^{-1}, \quad \text{i.e.,} \quad \mathbf{y}^T B = \mathbf{c}_{\mathcal{B}}^T,$$

$$\text{i.e.,} \quad [y_1 \quad y_2 \quad y_3] \begin{array}{c} \begin{array}{ccc} x_1 & x_4 & x_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} x_1 & x_4 & x_6 \end{array} \\ \begin{pmatrix} 3 & 0 & 0 \end{pmatrix} \end{array}.$$

Get $y_1 = 0$, $y_3 = 0$, $y_2 = 3/2$: $\mathbf{y}^T = [0 \quad 3/2 \quad 0]$.

(b) Write and scan vector

$$\mathbf{z}_{\mathcal{N}}^T = \mathbf{y}^T N - \mathbf{c}_{\mathcal{N}}^T = [0 \quad 3/2 \quad 0] \begin{array}{c} \begin{array}{ccc} x_2 & x_3 & x_5 \end{array} \\ \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 3 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_2 & x_3 & x_5 \end{array} \\ \begin{pmatrix} 2 & 4 & 0 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} x_2 & x_3 & x_5 \end{array} \\ \begin{pmatrix} -2 & 1/2 & 3/2 \end{pmatrix} \end{array}.$$

Select x_2 to enter (Bland's Rule). Thus $E = 2$, $\mathbf{a}^{(E)} = \mathbf{a}^{(2)}$, $x_E = x_2$.

(2) Select Leaving Variable, noting $\mathbf{x}_{\mathcal{B}}(t) = \mathbf{x}_{\mathcal{B}}(0) - td$.

(a) Solve for $\mathbf{d}$ in

$$\mathbf{d} = B^{-1}\mathbf{a}^{(in)}, \quad \text{i.e.,} \quad B\mathbf{d} = \mathbf{a}^{(2)}$$

$$\text{i.e.,} \quad \begin{array}{c} \begin{array}{ccc} x_1 & x_4 & x_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \end{array} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Middle row gives $d_1 = 0$, then $d_2 = 1$ and $d_3 = 1$. So $\mathbf{d} = (0, 1, 1)$.

(b) Monitor

$$\mathbf{x}_{\mathcal{B}}(t) = \begin{array}{c} x_1 \\ x_4 \\ x_6 \end{array} \begin{pmatrix} 5/2 \\ 3/2 \\ 2 \end{pmatrix} - t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{array}{c} x_1 \\ x_4 \\ x_6 \end{array} \begin{pmatrix} 5/2 \\ 3/2 - t \\ 2 - t \end{pmatrix}$$

As t increases, $t^* = 3/2$ is the smallest value where $\mathbf{x}_{\mathcal{B}}(t)$ picks up a zero component. Variable x_4 labels that slot, so it will leave. Write $L = 4$, $x_L = x_4$, $\mathbf{a}^{(L)} = \mathbf{a}^{(4)}$.

(3) Update everything.

(a) New BFS refers to sets $\mathcal{B}$ and $\mathcal{N}$ above:

$$\mathbf{x}^*(t^*) = \begin{bmatrix} \mathbf{x}_{\mathcal{B}}(t^*) \\ \mathbf{x}_{\mathcal{N}}(t^*) \end{bmatrix} = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{pmatrix} 5/2 \\ 3/2 \\ 0 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}$$

(b) New basis selection sets $\mathcal{B} = \{1, 2, 6\}$ and $\mathcal{N} = \{3, 4, 5\}$.

(c) New partitioned matrices

$$B = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_6 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix} \end{array}, \qquad N = \begin{array}{c} \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \end{array},$$

$$\mathbf{c}_{\mathcal{B}}^T = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_6 \end{matrix} \\ \begin{pmatrix} 3 & 2 & 0 \end{pmatrix} \end{array}, \qquad \mathbf{c}_{\mathcal{N}}^T = \begin{array}{c} \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} 4 & 0 & 0 \end{pmatrix} \end{array},$$

$$\mathbf{x}_{\mathcal{B}}^* = B^{-1}\mathbf{b} = \begin{array}{c} x_1 \\ x_2 \\ x_6 \end{array} \begin{pmatrix} 5/2 \\ 3/2 \\ 1/2 \end{pmatrix}, \qquad f^* = 25/2.$$

Third iteration:

(1) Select entering variable.

(a) Solve for $\mathbf{y}$ in

$$\mathbf{y}^T = \mathbf{c}_{\mathcal{B}}^T B^{-1}, \quad \text{i.e.,} \quad \mathbf{y}^T B = \mathbf{c}_{\mathcal{B}}^T,$$

$$\text{i.e.,} \quad [y_1 \quad y_2 \quad y_3] \begin{array}{c} \begin{matrix} x_1 & x_2 & x_6 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_6 \end{matrix} \\ \begin{pmatrix} 3 & 2 & 0 \end{pmatrix} \end{array}.$$

Get $y_3 = 0$, then $y_1 = 2$, then $y_2 = 1/2$: $\mathbf{y}^T = [2 \quad 1/2 \quad 0]$.

(b) Write and scan vector

$$\mathbf{z}_{\mathcal{N}}^T = \mathbf{y}^T N - \mathbf{c}_{\mathcal{N}}^T = [2 \quad 1/2 \quad 0] \begin{array}{c} \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} 4 & 0 & 0 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} 3/2 & 2 & 1/2 \end{pmatrix} \end{array}.$$

All components are positive, so the current BFS is the UNIQUE MAXIMIZER! To report, recall the full solution vector $\mathbf{x}^*(t^*)$ shown explicitly in part 3(a) of the Second Iteration above. ////