[12] **1.** This city needs to show more support for its professional athletes, and I need to make some money. Here's a win-win proposal: I'll buy 30 bags of fibre, 80 hours of unskilled labour, and 20 hours of expert labour, and use these resources to make hats, jerseys, and scarves in the team colours. My manufacturing plan is in the table below:

Resource:	per hat:	per jersey:	per scarf:
Fibre (bag)	1	1	2
Unskilled Labour (hr)	3	3	2
Expert Labour (hr)	1	1	0

My net profit after expenses is \$6 for each hat, \$16 for each jersey, and \$8 for each scarf.

(a) Set up and solve a linear programming problem to determine how many hats, jerseys, and scarves I should make to maximize my profit.

Let $x_1 = \#\text{hats}$, $x_2 = \#\text{jerseys}$, $x_3 = \#\text{scarves}$. My LP:

$$\begin{aligned} \max \quad & f = 6x_1 + 16x_2 + 8x_3 \\ \text{s.t.} \quad & 0 \le w_1 = 30 - x_1 - x_2 - 2x_3 \\ & 0 \le w_2 = 80 - 3x_1 - 3x_2 - 2x_3 \\ & 0 \le w_3 = 20 - x_1 - x_2 \end{aligned}$$

and $\vec{x} \geq \vec{0}$ in $\mathbb{R}^3$.

PIVOT #1: $x_2 \leftrightarrow w_3$, via $x_2 = 20 - x_1 - w_3$.

$$f = 6x_1 \cancel{+16x_2} + 8x_3$$
$$320 - 16x_1 \qquad -16w_3$$
$$\Rightarrow \quad f = 320 - 10x_1 + 8x_3 - 16w_3$$

$$w_1 = 30 - x_1 \cancel{-x_2} - 2x_3$$
$$-20 + x_1 \qquad + w_3$$

$$w_2 = 80 - 3x_1 \cancel{-3x_2} - 2x_3$$
$$-60 + 3x_1 \qquad + 3w_3$$

$$\begin{aligned} x_2 &= 20 - x_1 - w_3 \\ w_1 &= 10 - 2x_3 + w_3 \\ w_2 &= 20 - 2x_3 + 3w_3 \end{aligned}$$

PIVOT #2: $x_3 \leftrightarrow w_1$, via $x_3 = 5 - \frac{1}{2}w_1 + \frac{1}{2}w_3$:

$$f = 320 - 10x_1 \cancel{+8x_3} - 16w_3$$
$$40 - 4w_1 \qquad + 4w_3$$
$$\Rightarrow \quad f = 360 - 10x_1 - 4w_1 - 12w_3$$

$$w_2 = 20 \cancel{-2x_3} + 3w_3$$
$$-10 \qquad + w_1 - w_3$$
$$\Rightarrow$$

$$\begin{aligned} x_2 &= 20 - x_1 - w_3 \\ x_3 &= 5 - \tfrac{1}{2}w_1 + \tfrac{1}{2}w_3 \\ w_2 &= 10 + w_1 + 2w_3 \end{aligned}$$

Continued on page 3

Unique maximizer: earn $f_{MAX} = 360$ (dollars)

by making $x_1 = 0$ hats

$x_2 = 20$ jerseys

$x_3 = 5$ scarves

[Note that $w_1 = 0$ bags of fibre are unused,
$w_2 = 10$ hours of unskilled labour are unused,
$w_3 = 0$ hours of skilled labour are unused.]

(b) Write and solve the dual problem for the profit-maximization problem in (a).

$$\begin{aligned}
\min \quad g &= 30y_1 + 80y_2 + 20y_3 \\
\text{s.t.} \quad & y_1 + 3y_2 + y_3 \geq 6 \\
& y_1 + 3y_2 + y_3 \geq 16 \\
& 2y_1 + 2y_2 \geq 8 \\
& \vec{y} \geq \vec{0} \text{ in } \mathbb{R}^3
\end{aligned}$$

Read from primal optimal dict, coeffs of w_1, w_2, w_3:

$\vec{y}^* = (4, 0, 12)$.

Ck: $g(\vec{y}^*) = 360 = f_{MAX}$, OK.

(c) Assuming I can sell everything I make, increasing my resources will obviously increase my profits. How much would it make sense to pay for a little more fibre? A little more unskilled labour? A little more expert labour? (Give units for your answers; do not insist on integer values for the commodities produced.)

This is what $\vec{y}^*$ is good for. My top bid for these resources is

$y_1^* = \$4/\text{bag}$ for fibre

$y_2^* = \$0/\text{hr}$ for unskilled labour (have too much already)

$y_3^* = \$12/\text{hr}$ for expert labour.

Continued on page 4

[12] **2.** Professor White has assigned the following problem:

(D)

$$\begin{array}{lrcrcrcl} \text{minimize } g = & 8y_1 & + & 16y_2 & + & 12y_3 & & \\ \text{subject to} & 2y_1 & + & 5y_2 & + & 4y_3 & \geq & 9 \\ & -y_1 & + & 3y_2 & + & y_3 & \geq & 3 \\ & 2y_1 & + & y_2 & - & y_3 & \geq & 5 \\ & 6y_1 & + & 2y_2 & + & 3y_3 & \geq & 22 \\ & & & y_1, y_2, y_3 \geq 0 & & & & \end{array}$$

Now three of her students are disputing which vector $\mathbf{y}^*$ gives the minimum.

- Doc says, "Trust me, I'm in pre-med. It's $\mathbf{y}^* = (2, 1, 0)$."
- Grumpy says, "Pivoting with fractions? Humbug! It took me an hour to get $\mathbf{y}^* = (3, 2, 0)$."
- Sleepy yawns and says, "I stayed up late working on this. The answer is $\mathbf{y}^* = (1, 5, 2)$."

Which of these students, if any, has the optimal vector? How can you tell?
(A *mathematically justified response* is required.)

- Test $g_{DOC} = 32$, $g_{GRUMPY} = 56$, $g_{SLEEPY} = 112$.
- Best g-value comes from DOC, but $\vec{y} = (2,1,0)$ fails inequality #2. So DOC is disqualified.
- Try GRUMPY: call the given problem (D), as it's the dual of

$$\begin{array}{llrcrcrcrcl} (P) & \max & f = 9x_1 & + & 3x_2 & + & 5x_3 & + & 22x_4 & & \\ & \text{s.t.} & 2x_1 & - & x_2 & + & 2x_3 & + & 6x_4 & \leq & 8 \\ & & 5x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & \leq & 16 \\ & & 4x_1 & + & x_2 & - & x_3 & + & 3x_4 & \leq & 12 \\ & & & & \vec{x} \geq \vec{0} & \text{in} & \mathbb{R}^4. & & & & \end{array}$$

(i) With $\vec{y} = (3,2,0)$, dual surplus vals are

$z_1 = 16 - 9 = 7$, $z_2 = 3 - 3 = 0$, $z_3 = 8 - 5 = 3$, $z_4 = 22 - 22 = 0$.

(ii)(iii) Complementarity: $y_1 > 0 \Rightarrow w_1 = 0$, $y_2 > 0 \Rightarrow w_2 = 0$, $z_1 > 0 \Rightarrow x_1 = 0$, $z_3 > 0 \Rightarrow x_3 = 0$.

So
$$\left.\begin{array}{l} 0 = w_1 = 8 \cancel{-2x_1} + x_2 \cancel{-2x_3} - 6x_4 \\ 0 = w_2 = 16 \cancel{-5x_1} - 3x_2 \cancel{-x_3} - 2x_4 \end{array}\right] \Rightarrow \left[\begin{array}{l} 6x_4 - x_2 = 8 \\ 2x_4 + 3x_2 = 16 \end{array}\right.$$

$x_1 = 0$ $\quad$ $x_3 = 0$

Continued on page 5

[12] **2.** Professor White has assigned the following problem:

$$\begin{array}{ll} \text{minimize } g = & 8y_1 + 16y_2 + 12y_3 \\ \text{subject to} & 2y_1 + 5y_2 + 4y_3 \geq 9 \\ & -y_1 + 3y_2 + y_3 \geq 3 \\ & 2y_1 + y_2 - y_3 \geq 5 \\ & 6y_1 + 2y_2 + 3y_3 \geq 22 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Now three of her students are disputing which vector $\mathbf{y}^*$ gives the minimum.

- Doc says, "Trust me, I'm in pre-med. It's $\mathbf{y}^* = (2, 1, 0)$."
- Grumpy says, "Pivoting with fractions? Humbug! It took me an hour to get $\mathbf{y}^* = (3, 2, 0)$."
- Sleepy yawns and says, "I stayed up late working on this. The answer is $\mathbf{y}^* = (1, 5, 2)$."

Which of these students, if any, has the optimal vector? How can you tell?
(A *mathematically justified response* is required.)

This little 2x2 system has unique solution

$x_2 = 4, \quad x_4 = 2.$

Thus (i)-(iii) hold for $\vec{y}^* = (3,2,0)$ with $\vec{x} = (0,4,0,2)$.

(iv) We already enforced $w_1 = 0$, $w_2 = 0$, so it remains only to check $w_3 = 12 - (10) = 2 > 0$. This shows that $\vec{x}$ is primal-feasible.

Optimality of $\vec{y}^* = (3,2,0)$ is confirmed. GRUMPY WINS!

(Sleepy's value is too large, so he's no longer in contention.)

[Note: Grumpy's $\vec{x} = (0,4,0,2)$ gives the primal value $f = 56 = g(\vec{y}^*)$, as it should.]

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[12] **3.** (a) Introduce slack variables and write the corresponding (infeasible) initial dictionary for this problem:

$$\begin{array}{ll} \text{maximize } f = & -6x_1 - 8x_2 - 4x_3 - 9x_4 \\ \text{subject to} & x_1 + 2x_2 - x_3 + x_4 \geq 0 \longleftrightarrow -x_1 - 2x_2 + x_3 - x_4 \leq 0 \\ & -2x_1 - 4x_2 - x_3 + x_4 \leq -3 \\ & x_1 + 2x_3 \geq 2 \longleftrightarrow -x_1 - 2x_3 \leq -2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$\begin{array}{l} f = -6x_1 - 8x_2 - 4x_3 - 9x_4 \\ \hline w_1 = x_1 + 2x_2 - x_3 + x_4 \\ w_2 = -3 + 2x_1 + 4x_2 + x_3 - x_4 \\ w_3 = -2 + x_1 + 2x_3 \end{array}$$

(b) Show how to use the Dual Simplex Method to solve the problem in part (a). Do no more than 3 pivots. (There is more space to write on the next page.)

PIVOT #1: Worst infeasibility is in w_2, so consider (for $t \geq 0$)

$$f + tw_2 = -3t - (6-2t)x_1 - (8-4t)x_2 - (4-t)x_3 - (9+t)x_4.$$

Smallest $t \geq 0$ to make a 0-coeff on right is $t=2$, selecting x_2.

So $x_2 \leftrightarrow w_2$, via $4x_2 = 3 - 2x_1 - x_3 + x_4 + w_2$

or $x_2 = \frac{3}{4} - \frac{1}{2}x_1 - \frac{1}{4}x_3 + \frac{1}{4}x_4 + \frac{1}{4}w_2$.

Now $\left.\begin{array}{l} f = -6x_1 \cancel{-8x_2} - 4x_3 - 9x_4 \\ \quad -6 + 4x_1 + 2x_3 - 2x_4 - 2w_2 \end{array}\right\} \Rightarrow f = -6 - 2x_1 - 2x_3 - 11x_4 - 2w_2$

$\left.\begin{array}{l} w_1 = x_1 \cancel{+2x_2} - x_3 + x_4 \\ \quad \frac{3}{2} - x_1 - \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}w_2 \end{array}\right\} \Rightarrow w_1 = \frac{3}{2} - \frac{3}{2}x_3 + \frac{3}{2}x_4 + \frac{1}{2}w_2$

$x_2 = \frac{3}{4} - \frac{1}{2}x_1 - \frac{1}{4}x_3 + \frac{1}{4}x_4 + \frac{1}{4}w_2$

$w_3 = -2 + x_1 + 2x_3$

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PIVOT #2: Worst infeasibility is in w_3, so consider (for $t \geq 0$)

$$f + t w_3 = -(6+2t) - (2-t)x_1 - (2-2t)x_3 - 11x_4 - 2w_2.$$

Smallest $t \geq 0$ to create a 0-coeff is $t=1$, sponsoring pivot $x_3 \leftrightarrow w_3$:

$$2x_3 = 2 - x_1 + w_3 \quad \Leftrightarrow \quad x_3 = 1 - \tfrac{1}{2}x_1 + \tfrac{1}{2}w_3.$$

Now
$$\left.\begin{aligned} f = {} & -6 - 2x_1 - 2x_3 - 11x_4 - 2w_2 \\ & -2 + x_1 \qquad\qquad\qquad\qquad - w_3 \end{aligned}\right\} \Rightarrow f = -8 - x_1 - 11x_4 - 2w_2 - w_3$$

$$\left.\begin{aligned} x_2 = {} & \tfrac{3}{4} - \tfrac{1}{2}x_1 - \tfrac{1}{4}x_3 + \tfrac{1}{4}x_4 + \tfrac{1}{4}w_2 \\ & -\tfrac{1}{4} + \tfrac{1}{8}x_1 \qquad\qquad\qquad - \tfrac{1}{8}w_3 \end{aligned}\right\} \Rightarrow x_2 = \tfrac{1}{2} - \tfrac{3}{8}x_1 + \tfrac{1}{4}x_4 + \tfrac{1}{4}w_2 - \tfrac{1}{8}w_3$$

$$x_3 = 1 - \tfrac{1}{2}x_1 + \tfrac{1}{2}w_3$$

$$\left.\begin{aligned} w_1 = {} & \tfrac{3}{2} \qquad - \tfrac{3}{2}x_3 + \tfrac{3}{2}x_4 + \tfrac{1}{2}w_2 \\ & -\tfrac{3}{2} + \tfrac{3}{4}x_1 \qquad\qquad - \tfrac{3}{4}w_3 \end{aligned}\right\} \Rightarrow w_1 = 0 + \tfrac{3}{4}x_1 + \tfrac{3}{2}x_4 + \tfrac{1}{2}w_2 - \tfrac{3}{4}w_3$$

WINNER! $\vec{x}^* = (0, \tfrac{1}{2}, 1, 0)$ is the UNIQUE MAXIMIZER.

(c) Write the problem that is dual to the one in part (a), and find all solutions for the dual.

(D) min $g = -3y_2 - 2y_3$

s.t.
$$\begin{aligned} -y_1 - 2y_2 - y_3 &\geq -6 \\ -2y_1 - 4y_2 &\geq -8 \\ y_1 - y_2 - 2y_3 &\geq -4 \\ -y_1 + y_2 &\geq -9 \\ \vec{y} \geq \vec{0} &\text{ in } \mathbb{R}^3 \end{aligned}$$

ONE dual solution is $\vec{y}^* = (0,2,1)$, from slack coeffs in optimal dict from (b). But is that all? Consider dual dict, from negative-transpose prop:

$$\begin{aligned} -g &= 8 - \tfrac{1}{2}z_2 - z_3 - 0y_1 \\ z_1 &= 1 + \tfrac{3}{8}z_2 + \tfrac{1}{2}z_3 - \tfrac{3}{4}y_1 \\ z_4 &= 11 - \tfrac{1}{4}z_2 \qquad\quad - \tfrac{3}{2}y_1 \\ y_2 &= 2 - \tfrac{1}{4}z_2 \qquad\quad - \tfrac{1}{2}y_1 \\ y_3 &= 1 + \tfrac{1}{8}z_2 - \tfrac{1}{2}z_3 + \tfrac{3}{4}y_1 \end{aligned}$$

Any $y_1 = t$, $0 \leq t \leq \tfrac{4}{3}$, gives a dual solution. Full list: $(t, 2 - \tfrac{t}{2}, 1 + \tfrac{3}{4}t)$.
[Segment endpts $(0,2,1)$ and $(\tfrac{4}{3}, \tfrac{4}{3}, 2)$.]

(c) Write the problem that is dual to the on in part (a), and $\boxed{\text{find all solutions}}$ for the dual.

ALTERNATIVE: Complementarity approach.

(i) $\vec{x}^* = (0,2,1) \geq \vec{0}$ OK, with slack vector $\vec{w}^* = (0,0,0) \geq \vec{0}$ OK.

(ii) $\vec{w}^* = (0,0,0)$ gives no usable info about $\vec{y}$.

(iii) $x_2^* > 0$, $x_3^* > 0$ force $z_2 = 0$ and $z_3 = 0$. That is,

$$\left.\begin{array}{l} 2y_1 + 4y_2 = 8 \\ -y_1 + y_2 + 2y_3 = 4 \end{array}\right] \Leftrightarrow \left[\begin{array}{l} y_2 = 2 - \frac{1}{2}y_1 \\ y_3 = 1 + \frac{3}{4}y_1 \end{array}\right.$$

(iv) For dual feasibility, enforce

$y_1 \geq 0$ | $z_1 \geq 0$ (so $6 - y_1 - 2y_2 - y_3 = 1 - \frac{3}{4}y_1 \geq 0$, i.e., $y_1 \leq \frac{4}{3}$)

$y_2 \geq 0$ (so $y_1 \leq 4$) | $z_2, z_3 \geq 0$ (automatic-enforced in (iii))

$y_3 \geq 0$ (so $y_1 \geq -\frac{4}{3}$) | $z_4 \geq 0$ (so $9 - y_1 + y_2 = 11 - \frac{3}{2}y_1 \geq 0$, i.e., $y_1 \leq \frac{22}{3}$)

All these hold simultaneously when $0 \leq y_1 \leq \frac{4}{3}$.

ANS: $\vec{y}^* = \left(y_1, 2 - \frac{y_1}{2}, 1 + \frac{3}{4}y_1\right)$, $0 \leq y_1 \leq \frac{4}{3}$, are the dual sol'ns.

[12] **4.** Use the graphical method outlined below to solve the following problem:

$$\left.\begin{array}{ll} \text{maximize } f = x_1 - 6x_2 - 4x_3 - 3x_4 & \\ \text{subject to} \quad x_1 - x_2 - x_3 - x_4 \le -1 & \\ \qquad\qquad\quad x_1 - 2x_2 - x_3 \qquad \le -1 & \\ \qquad\qquad\quad x_1, x_2, x_3, x_4 \ge 0 & \end{array}\right] \quad (P)$$

(a) Write out problem (D), the dual of (P).

(b) Make a good sketch of the set of all feasible input vectors $\mathbf{y}$ for (D).

(c) Using your sketch from (b), or otherwise, find an optimal feasible point for (D).

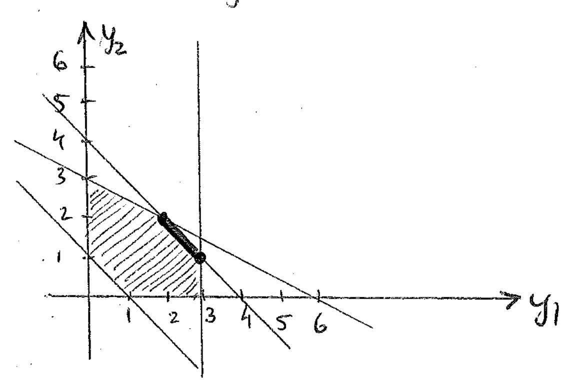
(d) Use complementary slackness with the result from (c) to find all solutions for (P).

(a) $\min\ g = -y_1 - y_2$ $\Leftrightarrow$ $-\max(-g) = y_1 + y_2$

s.t.
$$\begin{aligned} y_1 + y_2 &\ge 1 \\ -y_1 - 2y_2 &\ge -6 \\ -y_1 - y_2 &\ge -4 \\ -y_1 \quad &\ge -3 \\ \vec{y} \ge \vec{0} &\text{ in } \mathbb{R}^2 \end{aligned}$$

s.t.
$$\begin{aligned} y_1 + y_2 &\ge 1 \\ y_1 + 2y_2 &\le 6 \\ y_1 + y_2 &\le 4 \\ y_1 \quad &\le 3 \\ y_1 \ge 0, &\ y_2 \ge 0 \end{aligned}$$

(b)



Dual-feasible pt $\vec{y}$	$g(\vec{y})$
$(2,2)$	-4
$(3,1)$	-4

(c) maximizing $(-g) = (1,1)\cdot\vec{y}$ over polygon in sketch selects the whole line segment joining $(2,2)$ to $(3,1)$. Any point on that segment will work in what follows.

Continued on page 8

(Blank page for extra calculations.)

(d) With $\vec{y}^* = (2,2)$, dual surplus $\vec{z} = (3,0,0,1)$.

Complementarity gives $x_1 = 0, x_4 = 0, w_1 = 0, w_2 = 0$, i.e.,

$$\left.\begin{aligned} x_2 + x_3 &= 1 \\ 2x_2 + x_3 &= 1 \end{aligned}\right\} \Rightarrow x_2 = 0, x_3 = 1.$$

This is primal-feasible, hence optimal. $\vec{x}^* = (0,0,1,0)$.

ALT: with $\vec{y}^* = (3,1)$, dual surplus $\vec{z} = (3,1,0,0)$.

Complementarity gives $x_1 = 0, x_2 = 0, w_1 = 0, w_2 = 0$, i.e.,

$$\left.\begin{aligned} x_3 + x_4 &= 1 \\ x_3 \qquad &= 1 \end{aligned}\right\} \Rightarrow x_3 = 1, x_4 = 0.$$

This is primal-feasible, hence optimal. Again, $\vec{x}^* = (0,0,1,0)$.

ALT: With $\vec{y}^* = (2+t, 2-t)$, some t with $0 < t < 1$,
dual surplus $\vec{z} = (3, t, 0, 1-t)$ has 3 positive entries.
Hence $x_1 = 0, x_2 = 0, x_4 = 0$ along with $w_1 = 0, w_2 = 0$ for complementarity, leading to the pair of eq^{n}s

$$x_3 = 1 \quad \text{AND} \quad x_3 = 1.$$

This leads again to $\vec{x}^* = (0,0,1,0)$ ... primal-feasible, hence optimal.

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[12] **5.** Prove that it is impossible to choose constants a, b, c, d so that the following problem has a maximizer:

(P)

$$\begin{array}{ll} \text{maximize } f = & 10x_1 - x_2 - 8x_3 \\ \text{subject to} & 2x_1 - 2x_2 - x_3 \le a \\ & -x_1 + 2x_2 - 2x_3 \le b \\ & -x_1 \qquad - 2x_3 \le c \\ & -2x_1 - x_2 + x_3 \le d \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

A clear and complete explanation is required for full credit. Clearly state ~~about~~ any famous facts or theorems that you use.

Hint: Write the dual problem and add its constraints.

Dual Problem:

$$\begin{array}{ll} \min & g = ay_1 + by_2 + cy_3 + dy_4 \\ \text{s.t.} & 2y_1 - y_2 - y_3 - 2y_4 \geq 10 \\ & -2y_1 + 2y_2 \qquad - y_4 \geq -1 \\ & -y_1 - 2y_2 - 2y_3 + y_4 \geq -8 \\ & \vec{y} \geq \vec{0} \text{ in } \mathbb{R}^4 \end{array}$$

If $\vec{y} \in \mathbb{R}^4$ is feasible in (D), all 3 inequalities above must be valid simultaneously. Adding them then implies

$$-y_1 - y_2 - 3y_3 - 2y_4 \geq 1.$$

But $\vec{y} \geq \vec{0}$ componentwise, so this is impossible.

Hence <u>PROBLEM (D) IS INFEASIBLE.</u> In shorthand, $\min(D) = +\infty$.

Two possibilities arise:

① Extreme Duality Gap, ie., $\max(P) = -\infty$.
Here (P) IS ALSO INFEASIBLE, so no maximizer exists

② Equal Primal/Dual Values, ie., $\max(P) = +\infty$.
Here (P) is UNBOUNDED, so no maximizer exists.