*This final exam has **7 questions** on **15 pages**, for a total of 80 marks.*

Duration: 2 hours 30 minutes

Full Name (including all middle names): ______________________________

Student-No: ______________________________

Signature: ______________________________

UBC Rules governing examinations:

1. Each candidate should be prepared to produce his/her library/AMS card upon request.
2. No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour or the *last 15 minutes* of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.
3. Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

   a) Making use of any books, papers or memoranda, other than those authorised by the examiners.

   b) Speaking or communicating with other candidates.

   c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness will not be received.
4. Smoking is not permitted during examinations.

Question:	1	2	3	4	5	6	7	Total
Points:	10	16	16	12	10	8	8	80
Score:								

Please read the following points carefully before starting to write.

- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.). **If you have any of these then tell me now!**
- You may not leave during the first 30 minutes or final 30 minutes of the exam.
- Read all the questions carefully before starting to work.
- Continue on the back of the previous page if you run out of space.
- When choosing entering and leaving variables remember to use Anstee's rules
  - use the entering variable with the largest positive coefficient, and if there is a choice then pick the one with the smallest subscript.
  - if there is a choice of leaving variable, then choose the one with the smallest subscript.
- The following formulae may be useful. You are assumed to understand what they mean.

$$\begin{aligned} \vec{x}_B &= \mathbf{B}^{-1}\vec{b} - \mathbf{B}^{-1}\mathbf{A}_N\vec{x}_N \\ z &= \vec{c}_B^T\mathbf{B}^{-1}\vec{b} + \left(\vec{c}_N^T - \vec{c}_B^T\mathbf{B}^{-1}\mathbf{A}_N\right)\vec{x}_N \end{aligned}$$

$\boxed{\text{10 marks}}$ 1. (a) What is a degenerate dictionary? Give an example.

(b) What is a "feasible origin"?

(c) What is the smallest subscript rule? Explain (briefly) why it is important.

(d) Carefully state the strong duality theorem.

16 marks 2. Use the revised simplex method to find a solution of the following LP problem (remember to use Anstee's rule).

$$\begin{array}{llrrrl}
\text{Maximise} & z = & 3x_1 & -3x_2 & +2x_3 & \\
\text{Subject to} & & 2x_1 & -2x_2 & +3x_3 & \leq 4 \\
 & & 2x_1 & -3x_2 & +2x_3 & \leq 3 \\
 & & -x_1 & +2x_2 & +x_3 & \leq 6 \\
 & & x_1, & x_2, & x_3 & \geq 0
\end{array}$$

Clearly state the relevant matrices (with appropriate basis headings) at each iteration.

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16 marks 3. You have decided to start the Oppel rubber company. You are going to make two exciting new products: new footwear called the "iMoc" and a fruit protector called the "oPod". Each iMoc brings \$9 of profit and each oPod brings \$6.

You have 18 hours of labour available per day and 24 units of rubber. Each iMoc requires 2 units of rubber and 1 hour of labour while each oPod requires 1 unit of rubber and 2 hours of labour. Additionally each oPod must be rigorously tested with real bananas and you can buy 6 per day.

Let x_1 be the number of iMocs and let x_2 be the number of oPods produced each day. The corresponding LP problem is

$$\begin{array}{lrrrl} \text{Maximise} & z = & 9x_1 & +6x_2 & \\ \text{Subject to} & & x_1 & +2x_2 & \leq 18 \\ & & 2x_1 & +x_2 & \leq 24 \\ & & & x_2 & \leq 6 \\ & & x_1, & x_2 & \geq 0 \end{array}$$

(a) What is the dual of this problem?

(b) The primal LP is thought to have an optimal solution when x_1, x_2, x_5 are basic and x_3, x_4 are non-basic. What is the basic feasible solution given this choice of basic and non-basic variables? Give the values of all decision and slack variables and the objective function.

(c) Use complementary slackness to confirm that this solution is indeed optimal.

(d) What are the optimal values of the dual variables?

(e) What are the units of the dual variables y_1, y_2, y_3?

(f) Using some additional funds you are able to purchase one more unit of rubber *or* fund one more hour of labour *or* buy one more banana. Given that these changes are small, what should you do?

12 marks 4. Consider the following LP.

$$
\begin{array}{lrrrrl}
\text{Maximise} & z = & 2x_1 & +3x_2 & +4x_3 & \\
\text{Subject to} & & x_1 & +2x_2 & +3x_3 & \leq 7 \\
 & & 2x_1 & +2x_2 & +5x_3 & \leq 8 \\
 & & & 2x_2 & +3x_3 & \leq 5 \\
 & & x_1, & x_2, & x_3 & \geq 0
\end{array}
$$

The optimal dictionary is obtained when x_1, x_2 and x_4 are basic and x_3, x_5 and x_6 are non-basic. In this case

$$
\mathbf{B} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \end{array} \qquad \mathbf{B}^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{bmatrix}
$$

(a) If the second constraint is replaced by

$$2x_1 + 2x_2 + 5x_3 \leq 8 + \beta$$

then over what range of β is the dictionary still optimal? What are the optimal solution and optimal value (as a function of β).

(b) Return to the original dictionary, and now replace the objective function by

$$z = (2 + \gamma)x_1 + 3x_2 + (4 + 3\gamma)x_3$$

Over what range of γ is the current dictionary optimal? In this range, what are the optimal solution and optimal value (as a function of γ)?

10 marks

5. Alice and Bob have decided to stop cryptographic experiments for the afternoon and are going to relax with a pot of tea and a matrix game. Alice is to be the row-player and Bob will be the column-player. Their friend Eve suggests a game with the following playoff matrix (to Alice)

$$\begin{bmatrix} 5 & -6 \\ -3 & 2 \end{bmatrix}$$

(a) What is Alice's optimal strategy? Use linear programming to solve this question.

(b) Is the game fair? If not, then who does it favour?

This page has been left blank for your workings and solutions.

8 marks 6. Consider the following LP problem and its optimal dictionary:

$$\begin{array}{llrrl} \text{Maximise} & z = & 6x_1 & +8x_2 & \\ \text{Subject to} & & 2x_1 & +x_2 & \leq 5 \\ & & 2x_1 & +3x_2 & \leq 11 \\ & & x_1, & x_2, & \geq 0 \end{array}$$

$$\begin{array}{ccrrr} x_1 & = & 4 & -3x_3 & +x_4 \\ x_2 & = & 1 & +2x_3 & -x_4 \\ \hline v & = & 32 & -2x_3 & -2x_4 \end{array}$$

Find the new optimal value and solution when the following constraint is added to the problem.

$$3x_1 + 2x_2 \leq 13$$

This page has been left blank for your workings and solutions.

8 marks 7. (a) Your colleague gives you an LP problem and tells you that they are having a small problem pivoting the current dictionary. It turns out that they can choose an entering variable, but there is no leaving variable. Explain (with a short proof) what this means about the problem.

(b) Paul and Carole are playing a matrix game. They have played several rounds and decided it is unfair — in particular it has value $q > 0$. Explain (with a short proof) why subtracting q from each element of the payoff matrix will make the game fair.