Be sure this exam has 5 pages.

THE UNIVERSITY OF BRITISH COLUMBIA

Sessional Examination - April 21 2006

MATH 340: Linear Programming

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Special Instructions: No calculators. You must show your work and **explain** your answers. Quote names of theorems used as appropriate. Time: 3 hours Total marks: 100

1. [13 marks]
   a) [10pts] Solve the following linear programming problem, using our standard two phase method and using Anstee's rule.

$$\begin{array}{lcccl} \text{Maximize} & x_1 & +2x_2 & & \\ & 2x_1 & +x_2 & -x_3 & \le 4 \\ & x_1 & -x_2 & & \le -3 \\ & x_1 & & -x_3 & \le -1 \end{array} \qquad x_1, x_2, x_3 \ge 0$$

   b) [3 marks] Give one more optimal solution (different from the one computed in a).

2. [10 marks] Consider the following linear program:

$$\begin{array}{lcccl} \text{Maximize} & -2x_1 & +4x_2 & -6x_3 & \\ & 2x_1 & +x_2 & -x_3 & \le 2 \\ & x_1 & +x_2 & +x_3 & \le 5 \\ & -x_1 & +2x_2 & -3x_3 & \le 3 \end{array} \qquad x_1, x_2, x_3 \ge 0$$

   a) [2 marks] Give the Dual Linear Program of the above Linear Program.
   b) [6 marks] You are given that an optimal dual solution has $x_1^* = 0$, $x_2^* = 3$, $x_3^* = 1$. Determine an optimal dual solution (without pivoting), stating which theorems you have used.
   c) [2 marks] Is the dual solution computed in b) degenerate? Does the dual solution remain optimal if we replace the objective function $-2x_1 + 4x_2 - 6x_3$ by the objective function $-2x_1 + 2x_2 - 7x_3$? Explain.

3. [10 marks]
   a) [8 marks] Given $A, \mathbf{b}, \mathbf{c}$, current basis (and B^{-1} for computational ease), use our Revised Simplex method to determine the next entering variable (if there is one), the next leaving variable (if there is one), and the new basic feasible solution after the pivot (if there is both an entering and leaving variable). The current basis is $\{x_3, x_1, x_4\}$.

$$\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 3 & 10 & 2 & 3 & 1 & 0 & 0 \\ 3 & 11 & 2 & 4 & 0 & 1 & 0 \\ 1 & 4 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \mathbf{b} \\ \begin{pmatrix} 7 \\ 7 \\ 3 \end{pmatrix} \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_3 \\ x_1 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} -1 & 0 & 3 \\ 2 & -1 & -2 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

$$\begin{array}{cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \mathbf{c} & (\,1 & 8 & 2 & 0 & 0 & 0 & 0\,) \end{array}$$

b) [2 marks] Is the solution associated with the basis $\{x_3, x_1, x_4\}$ degenerate? Why? Was the chosen pivot degenerate in a)? Why?

4. [25 marks] A manufacturer wishing to maximize profit can obtain three possible chairs made from the three available resources according to the following table.

	chair 1	chair 2	chair 3	availability
space	1	1	1	4
wood	2	3	3	10
labour	2	3	4	11
$ profit	6	8	9	

Let x_i denote the amount of chair i to produce and let x_{3+i} denote the ith slack for $i = 1, 2, 3$. The final dictionary is:

$$\begin{array}{rcllll} x_1 & = & 2 & -3x_4 & +x_5 & \\ x_2 & = & 1 & +2x_4 & -2x_4 & +x_6 \\ x_3 & = & 1 & & +x_5 & -x_6 \\ z & = & 29 & -2x_4 & -x_5 & -x_6 \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 3 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \end{array}$$

NOTE: All questions are independent of one another.

a) [2 marks] Give the marginal values for each of the resources space, wood and labour.

b) [5 marks] Give the range on b_2 (resource availability for wood) so that the current basis remains optimal. Also give the profit as a linear function of b_2 in that range.

c) [3 marks] Consider a new chair (say chair 4) with requirements 2 of space,2 of wood and 2 of labour and profit $7. Are you interested in producing this new chair. Explain.

d) [5 marks] Give the range on c_2 (profit for chair 2) so that the current solution remains optimal. Also give the profit as a linear function of c_2 in that range.

Hint for e),f): You need not complete all of the very final dictionary, merely the basis and the constants and the z row.

e) [5 marks] Given resource availabilities of $\begin{pmatrix} 4 \\ 10 \\ 9 \end{pmatrix}$, obtain (using the Dual Simplex method) the new optimal solution as well as the new marginal values.

f) [5 marks] Consider adding a new constraint $2x_1 + x_2 \leq 3$ to our original problem. Solve using the Dual Simplex method. Report the new solution as well as the new marginal values.

5. [15 marks] I wish to purchase dietary supplements to meet certain needs for Vitamin A, Vitamin E, Vitamin C and Calcium. The minimum requirements have been pre-adjusted to account for my existing diet of cinnamon buns.

	100 gm supp 1	100 gm supp 2	100 gm supp 3	minimum required
Vitamin A	2	3	2	47
Vitamin E	4	1	9	88
Vitamin C	20	100	40	1000
Calcium	1.3	2.7	1.8	15
cost/100gm $	$10.99	$12.99	$16.99	

I wish to select a mix of dietary supplements at minimum cost subject to the specified minimum amounts of vitamins and Calcium.

Each question below is independent. The LINDO input/output on this page and the next page will be useful. You can compute decimals to two digits; no more is required.

a) [3 marks] If the Vitamin C minimum requirement is reduced to 600, what is the change in the optimal mix of supplements and the cost of that mix?

b) [3 marks] I discover that I should double the minimum amount of Calcium (because Calcium is said not to be readily absorbable). What is the change in the optimal mix of supplements and the effect on the cost.

c) [4 marks] We are offered a new mega supplement with 10 units of Vitamin A, 10 units of Vitamin E, 100 units of Vitamin C and 10 units of calcium with a cost of $32.99 per 100 gms. Should I buy this new supplement?

d) [5 marks] What do you expect to happen to the cost of the supplement mix if the Vitamin A minimum is reduce from 87 to 86 and the Vitamin E minimum is increased from 88 to 89? Prove that your expectation is correct, in view of the LINDO output.

The input to LINDO was as follows. The constraints have been labeled to aid readability:

```
MIN 10.99 SUPP1 + 12.99 SUPP2 + 16.99 SUPP3
SUBJECT TO
VITAMINA) 2 SUPP1 + 3 SUPP2 + 2 SUPP3 > 47
VITAMINE) 4 SUPP1 + 1 SUPP2 + 9 SUPP3 > 88
VITAMINC) 20 SUPP1 + 100 SUPP2 + 40 SUPP3 > 1000
CALCIUM) 1.3 SUPP1 + 2.7 SUPP2 + 1.8 SUPP3 > 15
END
```

The following is the output from LINDO:

OBJECTIVE FUNCTION VALUE

266.4847

VARIABLE	VALUE	REDUCED COST
SUPP1	9.311111	0.000000
SUPP2	6.155556	0.000000
SUPP3	4.955555	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
VITAMINA)	0.000000	−3.039333
VITAMINE)	0.000000	−1.088667
VITAMINC)	0.000000	−0.027833
CALCIUM)	22.644444	0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
SUPP1	10.990000	1.002000	3.180698
SUPP2	12.990000	24.494999	2.505000
SUPP3	16.990000	7.198421	2.505000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
VITAMINA	47.000000	11.736842	9.744186
VITAMINE	88.000000	104.750000	31.857141
VITAMINC	1000.000000	335.200012	445.999969
CALCIUM	15.000000	22.644444	INFINITY

6. [9 marks] Consider a two person zero sum game whose payoff matrix for player 1 (the row player) is

$$A = \begin{pmatrix} 6 & 4 & 2 \\ 1 & 5 & 7 \end{pmatrix}$$

a) [2 marks] State the Linear Program that could be used to determine both the value of the game and an optimal strategy for player 1.

b) [2 marks] Considering the mixed strategy $(1/2, 1/2)^T$ for player 1, give the resulting lower bound on $v(A)$, the value of the game.

c) [5 marks] Given that $(1/2, 0, 1/2)^T$ is an optimal mixed strategy for player 2 (the column player), compute (and verify in some way) an optimal mixed strategy for player 1 (the row player).

7. [14 marks] Let $A = (a_{ij})$ be an $m \times n$ matrix such that $A > 0$, i.e. $a_{ij} > 0$ for each choice of i and j. Let $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$ be an $n \times 1$ vector with $\mathbf{c} > \mathbf{0}$ i.e. $c_j > 0$ for each choice of j. Let $\mathbf{b}$ be $m \times 1$ vector.

a) [4 marks] Show that there exists some $m \times 1$ vector $\mathbf{z}$ with $A^T\mathbf{z} \geq \mathbf{c}$.

b) [10 marks] In b), you may use the result of a) even if you did not prove it. Show that:

either

i) There exists an $\mathbf{x} \geq \mathbf{0}$ with $A\mathbf{x} = \mathbf{b}$

or

ii) There exists a $\mathbf{y}$ with $A^T\mathbf{y} \geq \mathbf{c}$ and $\mathbf{b} \cdot \mathbf{y} < 0$

but not both.

Name theorems used as you use them.

8. [8] Consider the following LP:

$$\begin{array}{lc} \max & \mathbf{c} \cdot \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} + \Delta\mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

Assume that B yields an optimal basis in the case the vector $\Delta\mathbf{b} = \mathbf{0}$. Also assume for two vectors $\mathbf{d}$ and $\mathbf{e}$, that basis B also yields an optimal basis in the cases $\Delta\mathbf{b} = \mathbf{d}$ and $\Delta\mathbf{b} = \mathbf{e}$. (perhaps LINDO gave you this information).

a) [4 marks] Show that B yields an optimal basis for the case $\Delta\mathbf{b} = \frac{1}{2}(\mathbf{d} + \mathbf{e})$.

b) [4 marks] Show that B yields an optimal basis for the case $\Delta\mathbf{b} = \mathbf{b}$.