Be sure this exam has 5 pages.

THE UNIVERSITY OF BRITISH COLUMBIA

Sessional Examination - June 16 2005

MATH 340: Linear Programming

Instructor: Dr. R. Anstee, section 921

Special Instructions: No calculators. You must show your work and **explain** your answers. Quote names of theorems used as appropriate. Time: 3 hours Total marks: 100

1. [13 marks]
   a) [10pts] Solve the following linear programming problem, using our standard two phase method and using Anstee's rule.

$$\begin{array}{lrrrl} \text{Maximize} & 5x_1 & +x_2 & -x_3 & \\ & 3x_1 & +x_2 & -x_3 & \le -2 \\ & x_1 & -x_2 & -2x_3 & \le -3 \\ & x_1 & & & \le 2 \end{array} \qquad x_1, x_2, x_3 \ge 0$$

   b) [3 marks] Give all optimal solutions. Explain (briefly) why these are all possible optimal solutions.

2. [10 marks] Consider the following linear program:

$$\begin{array}{lrrrl} \text{Maximize} & 5x_1 & & +12x_3 & \\ & x_1 & -x_2 & +2x_3 & \le 3 \\ & -x_1 & +x_2 & +5x_3 & \le 5 \\ & 2x_1 & +3x_2 & +7x_3 & \le 16 \end{array} \qquad x_1, x_2, x_3 \ge 0$$

   a) [2 marks] Give the Dual Linear Program of the above Linear Program.
   b) [6 marks] You are given that an optimal dual solution has $y_1^* = 3$, $y_2^* = 0$, $y_3^* = 1$. Determine an optimal primal solution, stating which theorems you have used.
   c) [2 marks] Does the primal solution computed in b) remain optimal if we replace the objective function $5x_1 + 12x_3$ by the objective function $5x_1 + 13x_3$ and simultaneously replace the second inequality $-x_1 + x_2 + 5x_3 \le 5$ by $-x_1 + x_2 + 3x_3 \le 5$? Explain.

3. [8 marks] Given $A, \mathbf{b}, \mathbf{c}$, current basis and B^{-1}, use our revised simplex method to determine the next entering variable (if there is one), the next leaving variable (if there is one), and the new basic feasible solution after the pivot (if there is both an entering and leaving variable). The current basis is $\{x_7, x_3, x_4\}$.

$$\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 0 & 0 \\ -1 & 3 & 1 & 2 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \mathbf{b} \\ \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix} \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_7 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ \mathbf{c} \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & 10 & 3 & 4 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

4. [24 marks] A manufacturer wishing to maximize profit can obtain three possible foods made from the three available ingredients according to the following table.

	food 1	food 2	food 3	availability
flour	1	2	3	5
sugar	2	3	6	8
eggs	2	4	7	12
profit	4	7	10	

Let x_i denote the amount of food i to produce and let x_{3+i} denote the ith slack for $i = 1, 2, 3$. The final dictionary is:

$$\begin{array}{rcrrrr} x_1 & = & 1 & -3x_3 & +3x_4 & -2x_5 \\ x_2 & = & 2 & & -2x_4 & +x_5 \\ x_6 & = & 2 & -x_3 & +2x_4 & \\ z & = & 18 & -2x_3 & -2x_4 & -x_5 \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \end{array}$$

NOTE: All questions are independent of one another.

a) [2 marks] Give the marginal values for each of the ingredients flour, sugar and eggs.

b) [3 marks] Give the range on c_3 (profit coefficient of food 3) so that the current solution remains optimal.

c) [6 marks] If the profits are changed from $(4, 7, 10)$ to $(4+p, 7+p, 10+p)$, determine the range on the parameter p so that the current solution remains optimal. In that range, give the profit as a function of p.

d) [3 marks] If the ingredient availabilities are changed to $\begin{bmatrix} 20 \\ 30 \\ 50 \end{bmatrix}$, determine the new optimal solution and its objective function value.

Hint for e),f): You need not complete all of the very final dictionary, merely the basis and the constants and the z row.

e) [5 marks] Given a fourth food (use variable x_7) with ingredient requirements $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$ and profit of 11 per unit, obtain the new optimal solution as well as the new marginal values.

f) [5 marks] Consider adding a new constraint $x_1 + x_2 \leq 2$ to our original problem. Solve using the Dual Simplex method. Report the new solution as well as the new marginal values.

5. [14 marks] We are running a factory and can produce products of three possible types from four types of parts as follows.

	product 1	product 2	product 3	available parts
part 1	3	5	2	286
part 2	4	6	2	396
part 3	5	8	3	440
part 4	4	7	4	396
profit $	21	35	15	

We wish to choose our product mix to obtain maximum profit subject both to the limitations on the inventory of available parts but also subject to the restriction that at most 50% of the number of produced products can be of one type (you may check that the constraints PROD1<50, PROD2<50, PROD3<50 force this requirement).

The LINDO input/output on this page and the next page will be useful for parts a),b),c).

a) [2 marks] What are the marginal values of the four parts?

b) [4 marks] Mr. Edison visits the factory and offers to make a remarkable new part that substitutes for any of the four parts and will only charge $2 for each of these new parts. Would you buy some? How many would you buy?

c) [4 marks] The market for product 1 crashes and the profit drops to that of product 3. Should you change your production?

d) [4 marks] Compute the marginal cost for the total parts that make up product 2 and compare with the profit for product 2 (Please let 1.545455 be $\frac{17}{11}$ for this calculation). Why aren't they equal?

The input to LINDO was as follows. The constraints have been labeled to aid readability:

```
MAX 21 PROD1 + 35 PROD2 + 15 PROD3
SUBJECT TO
PART1) 3 PROD1 + 5 PROD2 + 2 PROD3 < 286
PART2) 4 PROD1 + 6 PROD2 + 2 PROD3 < 396
PART3) 5 PROD1 + 8 PROD2 + 3 PROD3 < 440
PART4) 4 PROD1 + 7 PROD2 + 4 PROD3 < 396
PROD1<50) 0.5 PROD1 - 0.5 PROD2 - 0.5 PROD3 < 0
PROD2<50) - 0.5 PROD1 + 0.5 PROD2 - 0.5 PROD3 < 0
PROD3<50) - 0.5 PROD1 - 0.5 PROD2 + 0.5 PROD3 < 0
END
```

The following is the output from LINDO:

OBJECTIVE FUNCTION VALUE

1932.000

VARIABLE	VALUE	REDUCED COST
PROD1	22.000000	0.000000
PROD2	36.000000	0.000000
PROD3	14.000000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
PART1)	12.000000	0.000000
PART2)	64.000000	0.000000
PART3)	0.000000	3.000000
PART4)	0.000000	1.545455
PROD1<50)	14.000000	0.000000
PROD2<50)	0.000000	0.363636
PROD3<50)	22.000000	0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
PROD1	21.000000	0.363636	6.000000
PROD2	35.000000	INFINITY	0.500000
PROD3	15.000000	1.333333	2.615385

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
PART1	286.000000	INFINITY	12.000000
PART2	396.000000	INFINITY	64.000000
PART3	440.000000	24.000000	44.000000
PART4	396.000000	44.000000	23.692308
PROD1<50	0.000000	INFINITY	14.000000
PROD2<50	0.000000	22.000000	19.250000
PROD3<50	0.000000	INFINITY	22.000000

6. [9 marks] Consider a two person zero sum game whose payoff matrix for player 1 (the row player) is

$$A = \begin{pmatrix} 6 & 4 & 2 \\ 1 & 5 & 7 \end{pmatrix}$$

a) [2 marks] State the Linear Program that could be used to determine both the value of the game and an optimal strategy for player 1.

b) [2 marks] Considering the mixed strategy $(1/2, 1/2)^T$ for player 1, give the resulting lower bound on $v(A)$, the value of the game.

c) [5 marks] Given that $(1/2, 0, 1/2)^T$ is an optimal mixed strategy for player 2 (the column player), compute (and verify in some way) an optimal mixed strategy for player 1 (the row player).

7. [14 marks] Let $A = (a_{ij})$ be an $m \times n$ matrix such that $A > 0$, i.e. $a_{ij} > 0$ for each choice of i and j. Let $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$ be an $n \times 1$ vector with $\mathbf{c} > \mathbf{0}$ i.e. $c_j > 0$ for each choice of j. Let $\mathbf{b}$ be $m \times 1$ vector.

a) [4 marks] Show that there exists some $m \times 1$ vector $\mathbf{z}$ with $A^T\mathbf{z} \geq \mathbf{c}$.

b) [10 marks] In b), you may use the result of a) even if you did not prove it. Show that:

either

i) There exists an $\mathbf{x} \geq \mathbf{0}$ with $A\mathbf{x} = \mathbf{b}$

or

ii) There exists a $\mathbf{y}$ with $A^T\mathbf{y} \geq \mathbf{c}$ and $\mathbf{b} \cdot \mathbf{y} < 0$

but not both.

Name theorems used as you use them.

8. [8] Consider the following LP:

$$\begin{array}{rc} \max & \mathbf{c} \cdot \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} + \Delta\mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

Assume that B yields an optimal basis in the case the vector $\Delta\mathbf{b} = \mathbf{0}$. Also assume for two vectors $\mathbf{d}$ and $\mathbf{e}$, that basis B also yields an optimal basis in the cases $\Delta\mathbf{b} = \mathbf{d}$ and $\Delta\mathbf{b} = \mathbf{e}$. (perhaps LINDO gave you this information).

a) [4 marks] Show that B yields an optimal basis for the case $\Delta\mathbf{b} = \frac{1}{2}(\mathbf{d} + \mathbf{e})$.

b) [4 marks] Show that B yields an optimal basis for the case $\Delta\mathbf{b} = \mathbf{b}$.