Be sure this exam has 5 pages.

THE UNIVERSITY OF BRITISH COLUMBIA

Sessional Examination - June 12 2003

MATH 340: Linear Programming

Instructor: Dr. R. Anstee, section 921

Special Instructions: No calculators. You must show your work and **explain** your answers. Quote names of theorems used as appropriate. Time: 3 hours Total marks: 100

1. [12 marks]
   a) [10pts] Solve the following linear programming problem, using our standard two phase method and using Anstee's rule.

$$\begin{array}{rrrrl} \text{Maximize} & -x_1 & -x_2 & +x_3 & \\ & -x_1 & & -x_3 & \leq -1 \\ & -x_1 & -x_2 & & \leq -2 \\ & & -x_2 & +x_3 & \leq -1 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

   b) [2 marks] Find one additional optimal solution.

2. [11 marks] Consider the following linear program:

$$\begin{array}{rrrrl} \text{Maximize} & 12x_1 & +16x_2 & +2x_3 & \\ & 5x_1 & +2x_2 & +3x_3 & \leq 16 \\ & x_1 & +3x_2 & -x_3 & \leq 13 \\ & -x_1 & +x_2 & -3x_3 & \leq 0 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

   a) [2 marks] Give the Dual Linear Program of the above Linear Program.
   b) [7 marks] You are given that an optimal primal solution has $x_1^* = 0$, $x_2^* = 5$, $x_3^* = 2$. Determine an optimal dual solution, stating which theorems you have used.
   c) [2 marks] Does the primal solution given in a) remain optimal if we replace the objective function $12x_1 + 16x_2 + 2x_3$ by the objective function $14x_1 + 16x_2 + 2x_3$? By the objective function $12x_1 + 16x_2 + 3x_3$? Explain.

3. [10 marks]
   a) [8 marks] Given $A, \mathbf{b}, \mathbf{c}$, current basis and B^{-1}, use our revised simplex method to determine the next entering variable (if there is one), the next leaving variable (if there is one), and the new basic feasible solution after the pivot. The current basis is $\{x_7, x_2, x_3\}$.

$$\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array}\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array}\begin{array}{c} \mathbf{b} \\ \begin{pmatrix} -3 \\ -1 \\ -4 \end{pmatrix} \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_7 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ \mathbf{c} \end{array}\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 2 & -3 & 2 & 4 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

   b) [2 marks] Give a couple of good reasons why the revised simplex method (as implemented on computers) is important in practice.

4. [27 marks] A manufacturer wishing to maximize profit can obtain three possible products made from the three available raw materials of labour, energy and space according to the following table.

	product 1	product 2	product 3	availability
labour	2	1	3	10
energy	4	2	5	18
space	3	2	4	15
profit	6	3	8	

Let x_i denote the amount of product i to produce and let x_{3+i} denote the ith slack for $i = 1, 2, 3$. The final dictionary is:

$$\begin{array}{rcrrrr} x_1 & = & 1 & +2x_4 & -2x_5 & +x_6 \\ x_3 & = & 2 & -2x_4 & +x_5 & \\ x_2 & = & 2 & +x_4 & +x_5 & -2x_6 \\ z & = & 28 & -x_4 & -x_5 & \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_1 \\ x_3 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} -2 & 2 & -1 \\ 2 & -1 & 0 \\ -1 & -1 & 2 \end{pmatrix} \end{array}$$

a) [2 marks] Give the marginal values for labour, energy and space.

b) [5 marks] Give the range on c_3 (profit coefficient of product 3) so that the current solution remains optimal.

c) [6 marks] If the resource availabilities are changed to $\begin{pmatrix} 2p \\ 3p+3 \\ 3p \end{pmatrix}$, determine the range on the parameter p so that the current basis $\{x_1, x_3, x_2\}$ remains optimal. In that range, give the profit as a function of p. Note that $p = 5$ yields the original data.

Hint for d),e),f): You need not complete all of the very final dictionary, merely the basis and the constants and the z row.

d) [4 marks] Given a fourth product (use variable x_7) with raw material requirements $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and profit of 5 per unit, obtain the new optimal solution as well as the new marginal values.

e) [5 marks] Given that availabilities of labour, energy and space have changed to $\begin{pmatrix} 3 \\ 7 \\ 6 \end{pmatrix}$, predict the new profit using the marginal values and then apply the Dual Simplex method to obtain the new optimal solution as well as the new marginal values.

f) [5 marks] Consider adding a new constraint $x_1 + 2x_2 \leq 4$ to our original problem. Solve using the Dual Simplex method. Report the new solution as well as the new marginal values.

5. [17 marks] We wish to choose between three dietary supplements to achieve certain dietary needs of 4 vitamins and minerals at minimum cost. Imagine that the supplements are in liquid form so that you don't have to worry about choosing an integral amount of each supplement. We focus on 4 dietary needs: Vitamin E, Vitamin B12, Iron and Vitamin A. We have listed the contents of a millilitre of each supplement and its cost per millilitre.

	supplement 1	supplement 2	supplement 3	diet needs
vitamin E	1.0	0.1	0.4	5
vitamin B12	0.1	1.0	0.4	7
iron	0.1	0.1	0.4	3
vitamin A	0.1	0.1	0.2	2
cost per ml	10.2	11.3	14	

The LINDO output on the next page will be useful for parts a),b),c).

a) [4 marks] What are the marginal costs of Vitamin E, Vitamin B12, Iron and Vitamin A? Predict the cost if, because you are on a special diet for an extreme snowboarding trip to Everest, your dietary needs for Vitamin E, Vitamin B12, Iron and Vitamin A are now $(6, 10, 3, 3)$ respectively.

b) [4 marks] Why might we say that Supplement 1 is Vitamin E? Does its unit cost correspond to the marginal value of Vitamin E? What is in fact the most important ingredient of Supplement 1? Explain or comment sensibly.

c) [4 marks] Think of yourself as an owner of a health shop. How much could you raise the price of Supplement 1 and still sell the same amount?

d) [5 marks] Give a linear inequality (suitable for LINDO) that expresses the requirement that at least 20% of the volume of the purchased supplements are supplement 1. Do not attempt to solve.

The input to LINDO was as follows. Please note that this a minimization problem with inequalities $\geq$ entered as >. One consequence is that the dual prices are reported with a negative value which you must interpret in context. The constraints have been labeled to aid readability:

min 10.2supp1+11.3supp2+14supp3
st
vite) supp1+.1supp2+.4supp3 > 5
vitb12) .1supp1+1supp2+.4supp3 > 7
iron) .1supp1+.1supp2+.4supp3 > 3
vita) .1supp1+.1supp2+.2supp3 > 2
end

The following is the output from LINDO:

OBJECTIVE FUNCTION VALUE

162.6508

VARIABLE	VALUE	REDUCED COST
SUPP1	1.746032	0.000000
SUPP2	3.968254	0.000000
SUPP3	7.142857	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
VITE)	0.000000	−4.746032
VITB12)	0.000000	−5.968254
IRON)	0.428571	0.000000
VITA)	0.000000	−48.571430

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
SUPP1	10.200000	17.000000	3.737500
SUPP2	11.300000	17.000000	4.700000
SUPP3	14.000000	6.644444	6.181818

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
VITE	5.000000	3.000000	1.375000
VITB12	7.000000	3.000000	3.125000
IRON	3.000000	0.428571	INFINITY
VITA	2.000000	0.611111	0.166667

6. [10 marks] Let A be an $m \times n$ matrix, B be an $p \times n$ matrix, $\mathbf{c}$ and $\mathbf{x}$ be $n \times 1$ vectors, $\mathbf{y}$ be an $m \times 1$ vector and $\mathbf{z}$ be an $p \times 1$ vector.

   Show that either

   i) There exists an $\mathbf{x}$ with $A\mathbf{x} \leq \mathbf{0}$, $B\mathbf{x} = \mathbf{0}$, and $\mathbf{c} \cdot \mathbf{x} > 0$,

   or

   ii) There exists a $\mathbf{y} \geq \mathbf{0}$ and a $\mathbf{z}$ with $A^T\mathbf{y} + B^T\mathbf{z} = \mathbf{c}$,

   but not both.

   Name theorems used as you use them.

7. [13 marks] Consider a two person zero sum game whose payoff matrix for player 1 (the row player) is

$$\begin{pmatrix} 3 & 3 \\ 2 & 8 \\ 5 & 2 \end{pmatrix}$$

   a) [4 marks] State the Linear Program used to determine both the value of the game and an optimal strategy for player 1.
   b) [4 marks] Show that $(0, 1/3, 2/3)^T$ is an optimal mixed strategy for player 1 by computing (and verifying in some way) an optimal mixed strategy for player 2 (the column player).
   c) [5 marks] Given any payoff matrix A, state the Linear Program used to determine both the value of the game and an optimal mixed strategy for player 1 (the row player) and show that the resulting Linear Program always has an optimal solution. Explain.