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Final Examination – 27 June 2013

Mathematics 340

Linear Programming

Closed book examination

Time: 150 minutes

Special Instructions:

To receive full credit, all answers must be supported with clear and correct derivations. No calculators, notes, or other aids are allowed.

Rules governing examinations

1		15
2		15
3		15
4		15
5		15
6		20
7		5
Total		100

[15] **1.** (a) Use a two-phase simplex method with Anstee's pivoting rules to solve the problem below.

$$\begin{array}{ll} \text{maximize} & f = 2x_1 - x_2 + x_3 \\ \text{subject to} & 2x_1 + x_2 + x_3 \ge 4 \\ & 2x_2 - x_3 \ge 3 \\ & x_1 + x_3 \le 2 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

$\iff$ $-2x_1 - x_2 - x_3 \le -4$, $-2x_2 + x_3 \le -3$

[Notes: Look carefully at the inequalities.There is more space to work on the next page.]

Classic Phase One Add x_0 to large side of each inequality, minimize x_0.

$$\begin{aligned} \tilde f &= -x_0 \\ \hline w_1 &= -4 + x_0 + 2x_1 + x_2 + x_3 \\ w_2 &= -3 + x_0 + 2x_2 - x_3 \\ w_3 &= 2 + x_0 - x_1 - x_3 \end{aligned}$$

Shrewd initial pivot: push x_0 into basis, curing all feasibility probs from $x_0 \leftrightarrow w_1$

$$x_0 = 4 - 2x_1 - x_2 - x_3 + w_1$$

Calc. $w_2 = -3 \cancel{+x_0} + 2x_2 - x_3$
$\quad +4 - 2x_1 - x_2 - x_3 + w_1$

$w_3 = 2 \cancel{+x_0} - x_1 - x_3$
$\quad 4 \quad -2x_1 - x_2 - x_3 + w_1$

$$\begin{aligned} \tilde f &= -4 + 2x_1 + x_2 + x_3 - w_1 \\ \hline x_0 &= 4 - 2x_1 - x_2 - x_3 + w_1 \\ w_2 &= 1 - 2x_1 + x_2 - 2x_3 + w_1 \\ w_3 &= 6 - 3x_1 - x_2 - 2x_3 + w_1 \end{aligned}$$

Anstee's rules: x_1 enters (largest coeff), via

$$x_1 = \tfrac{1}{2} + \tfrac{1}{2}x_2 - x_3 + \tfrac{1}{2}w_1 - \tfrac{1}{2}w_2$$

Calc. $\tilde f = -4 \cancel{+2x_1} + x_2 + x_3 - w_1$
$\quad 1 \quad + x_2 - 2x_3 + w_1 - w_2$

$w_3 = 6 \cancel{-3x_1} - x_2 - 2x_3 + w_1$
$\quad -\tfrac{3}{2} \quad - \tfrac{3}{2}x_2 + 3x_3 - \tfrac{3}{2}w_1 + \tfrac{3}{2}w_2$

$$\begin{aligned} \tilde f &= -3 + 2x_2 - x_3 - w_2 \\ \hline x_0 &= 3 - 2x_2 + x_3 + w_2 \\ x_1 &= \tfrac{1}{2} + \tfrac{1}{2}x_2 - x_3 + \tfrac{1}{2}w_1 - \tfrac{1}{2}w_2 \\ w_3 &= 9/2 - \tfrac{5}{2}x_2 + x_3 - \tfrac{1}{2}w_1 + \tfrac{3}{2}w_2 \end{aligned}$$

Now x_2 is the only eligible entering var:

$$x_2 = \tfrac{3}{2} + \tfrac{1}{2}x_3 + \tfrac{1}{2}w_2 - \tfrac{1}{2}x_0$$

Calc. $x_1 = \tfrac{1}{2} \cancel{+\tfrac{1}{2}x_2} - x_3 + \tfrac{1}{2}w_1 - \tfrac{1}{2}w_2$
$\quad +\tfrac{3}{4} \quad +\tfrac{1}{4}x_3 \quad +\tfrac{1}{4}w_2 - \tfrac{1}{4}x_0$

$w_3 = 9/2 \cancel{-5/2\,x_2} + x_3 - \tfrac{1}{2}w_1 + \tfrac{3}{2}w_2$
$\quad -\tfrac{15}{4} \quad -\tfrac{5}{4}x_3 \quad -\tfrac{5}{4}w_2 + \tfrac{5}{4}x_0$

$$\begin{aligned} \tilde f &= -x_0 \\ \hline x_1 &= 5/4 - 3/4\,x_3 + \tfrac{1}{2}w_1 - \tfrac{1}{4}w_2 - \tfrac{1}{4}x_0 \\ x_2 &= 3/2 + 1/2\,x_3 + 1/2\,w_2 - \tfrac{1}{2}x_0 \\ w_3 &= 3/4 - 1/4\,x_3 - \tfrac{1}{2}w_1 + 1/4\,w_2 + \tfrac{5}{4}x_0 \end{aligned}$$

PHASE ONE COMPLETE! DROP VAR x_0, KEEP DICT, RESTORE f...

$$\left.\begin{array}{l} f = 2x_1 - x_2 + x_3 \\ \quad \frac{5}{2} \qquad -\frac{3}{2}x_3 + w_1 - \frac{1}{2}w_2 \\ \quad -\frac{3}{2} \qquad -\frac{1}{2}x_3 \qquad -\frac{1}{2}w_2 \end{array}\right] \Rightarrow$$

$$\begin{array}{rcl} f &=& 1 - x_3 + w_1 - w_2 \\ \hline x_1 &=& 5/4 - \frac{3}{4}x_3 + \frac{1}{2}w_1 - \frac{1}{4}w_2 \\ x_2 &=& 3/2 + \frac{1}{2}x_3 \qquad\qquad + \frac{1}{2}w_2 \\ w_3 &=& 3/4 - \frac{1}{4}x_3 - \frac{1}{2}w_1 + \frac{1}{4}w_2 \end{array}$$

Now w_1 enters, via

$$w_1 = \frac{3}{2} - \frac{1}{2}x_3 + \frac{1}{2}w_2 - 2w_3:$$

Calc $f = 1 - x_3 \cancel{+w_1} - w_2$

$\qquad 3/2 \; -\frac{1}{2}x_3 \qquad + \frac{1}{2}w_2 - 2w_3$

$x_1 = 5/4 - \frac{3}{4}x_3 \cancel{+\frac{1}{2}w_1} - \frac{1}{4}w_2$

$\qquad 3/4 - 1/4\, x_3 \qquad + \frac{1}{4}w_2 - w_3$

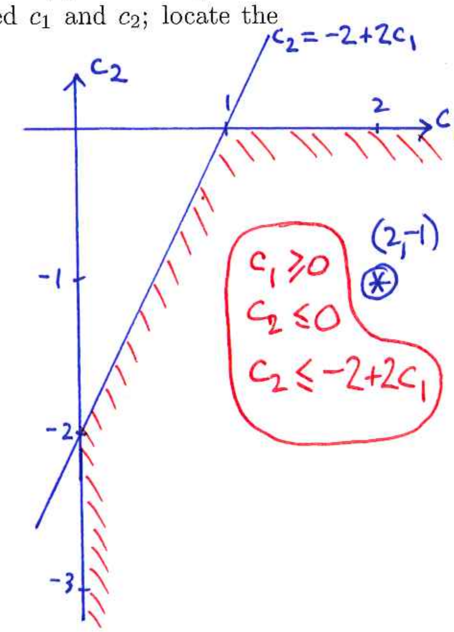
$$\begin{array}{rcl} f &=& 5/2 - 3/2\, x_3 - \frac{1}{2}w_2 - 2w_3 \\ \hline x_1 &=& 2 - x_3 \qquad\qquad - w_3 \\ x_2 &=& 3/2 + \frac{1}{2}x_3 + \frac{1}{2}w_2 \\ w_1 &=& 3/2 - \frac{1}{2}x_3 + \frac{1}{2}w_2 - 2w_3 \end{array}$$

** UNIQUE MAXIMIZER: $\vec{x}^* = (2, 3/2, 0)$, $f^* = 5/2$. **

(b) Suppose the objective function in the problem from part (a) is changed to

$$f = c_1x_1 + c_2x_2 + x_3.$$

Find all pairs (c_1, c_2) for which the basis identified in your solution for part (a) remains optimal. Sketch the set of all such pairs on a Cartesian plane with axes labelled c_1 and c_2; locate the special point $(c_1, c_2) = (2, -1)$ associated with part (a) on your sketch.



Sub
$$\begin{aligned} f &= c_1x_1 + c_2x_2 + x_3 \\ &= 2c_1 - c_1x_3 \qquad - c_1w_3 \\ &\quad + \frac{3}{2}c_2 + \frac{c_2}{2}x_3 + \frac{c_2}{2}w_2 \\ &\quad + x_3 \\ &= \left(2c_1 + \frac{3}{2}c_2\right) + \left(1 + \frac{c_2}{2} - c_1\right)x_3 + \frac{c_2}{2}w_2 - c_1w_3 \end{aligned}$$

Optimality requires simultaneous inequalities

$1 + \frac{c_2}{2} - c_1 \leq 0$, i.e., $c_2 \leq -2 + 2c_1$

$\frac{c_2}{2} \leq 0$, i.e., $c_2 \leq 0$

$-c_1 \leq 0$, i.e., $c_1 \geq 0$.

[When all these hold, $f_{MAX}(c_1, c_2) = 2c_1 + \frac{3}{2}c_2$.
This is consistent with (a) when $(c_1, c_2) = (2, -1)$.]

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[15] **2.** Given $h(x_1, x_2) = \min\{9 + x_1 + x_2, 10 + 2x_1 - x_2, 20 - 2x_1\}$, consider the optimization problem

$$(*) \qquad \overset{\text{max}}{\cancel{\text{min}}}\text{imize } h(x_1, x_2) \qquad \text{subject to} \quad x_1 \geq 0,\ x_2 \geq 0.$$

(a) Write a standard-form Linear Program equivalent to $(*)$. (Just write the LP: don't solve it yet!)

(b) Write the dual problem corresponding to the LP in part (a), and use Complementary Slackness to prove that the choices $x_1^* = 3$, $x_2^* = 2$ achieve the maximum in (a).

(c) Write the optimal dictionary for the problem in (a).

(d) Solve the following modification of problem $(*)$:

$$(**) \qquad \overset{\text{max}}{\cancel{\text{min}}}\text{imize } h(x_1, x_2) \qquad \text{subject to} \quad x_2 \leq \tfrac{1}{2},\ x_1 \geq 0,\ x_2 \geq 0.$$

(a) Trick shown in class for precisely this prob, re-used in Game Theory, is to invent new var x_3 to "push up" under the min:

$$\begin{aligned}
\text{maximize} \quad & x_3 \\
\text{subject to} \quad & x_3 \leq 9 + x_1 + x_2 \\
& x_3 \leq 10 + 2x_1 - x_2 \\
& x_3 \leq 20 - 2x_1 \\
& x_1 \geq 0,\ x_2 \geq 0.
\end{aligned}$$

Clearly the choice $x_1 = 0$, $x_2 = 0$, $x_3 = 9$ is feasible, so we can impose the constraint $x_3 \geq 0$ with no loss of generality. Std form:

$$(P) \left[\begin{aligned}
\text{maximize} \quad & x_3 \\
\text{subject to} \quad & -x_1 - x_2 + x_3 \leq 9 \\
& -2x_1 + x_2 + x_3 \leq 10 \\
& 2x_1 + x_3 \leq 20 \\
& \vec{x} \geq \vec{0} \text{ in } \mathbb{R}^3
\end{aligned}\right.$$

(b) DUAL PROB:

$$\left.\begin{aligned}
\text{minimize} \quad & 9y_1 + 10y_2 + 20y_3 \\
\text{s.t.} \quad & -y_1 - 2y_2 + 2y_3 \geq 0 \\
& -y_1 + y_2 \geq 0 \\
& y_1 + y_2 + y_3 \geq 1 \\
& \vec{y} \geq \vec{0} \text{ in } \mathbb{R}^3
\end{aligned}\right] \quad (D)$$

Test $x_1^* = 3$, $x_2^* = 2$; clearly $x_3^* = 14$

(i) In (P), slacks are $w_1^* = 0$, $w_2^* = 0$, $w_3^* = 0$. Primal feasibility ok.

(ii) Alternatives $w_j^* = 0$ OR $y_j^* = 0$ all hold from w_j^*. ok; nothing new.

(iii) Alternatives $x_j^* = 0$ OR $z_j^* = 0$ require

(1) $0 = z_1 = -y_1 - 2y_2 + 2y_3$

(2) $0 = z_2 = -y_1 + y_2$

(3) $0 = z_3 = y_1 + y_2 + y_3 - 1$

Get $y_2 = y_1$ from (2), then $2y_3 = y_1 + 2y_2 = 3y_1 \Rightarrow y_3 = \frac{3}{2} y_1$ from (1).

In (3), $1 = y_1 + y_2 + y_3 = y_1 + y_1 + \frac{3}{2} y_1 \Rightarrow y_1 = \frac{2}{7}$. Back-substitute

$y_1^* = \frac{2}{7}$, $y_2^* = \frac{2}{7}$, $y_3^* = \frac{3}{7}$.

(iv) Both $\vec{y}^* \geqslant \vec{0}$ in $\mathbb{R}^3$ and $\vec{z}^* \geqslant \vec{0}$ in $\mathbb{R}^3$, so mutual optimality is assured

(c) Write original dict defining primal slacks, then pivot by any convenient sequence to make x_1, x_2, x_3 basic. Or, invert a matrix.

$$x_4 = w_1 = 9 + x_1 + x_2 - x_3$$
$$x_5 = w_2 = 10 + 2x_1 - x_2 - x_3$$
$$x_6 = w_3 = 20 - 2x_1 - x_3$$

Try $x_2 = 10 + 2x_1 - x_3 - w_2$ for no fractions and simple x_6. Calc

$$w_1 = 9 + x_1 \cancel{+x_2} - x_3$$
$$10 + 2x_1 - x_3 - w_2$$

$$w_1 = 19 + 3x_1 - 2x_3 - w_2$$
$$x_2 = 10 + 2x_1 - x_3 - w_2$$
$$w_3 = 20 - 2x_1 - x_3$$

Next use $x_3 = 20 - 2x_1 - w_3$, no fractions:

$$w_1 = 19 + 3x_1 \cancel{-2x_3} - w_2$$
$$-40 + 4x_1 + 2w_3$$
$$x_2 = 10 + 2x_1 \cancel{-x_3} - w_2$$
$$-20 + 2x_1 + w_3$$

$$x_2 = -10 + 4x_1 - w_2 + w_3$$
$$w_1 = -21 + 7x_1 - w_2 + 2w_3$$
$$x_3 = 20 - 2x_1 - w_3$$

Finally, an ugly one is inevitable:

$$x_1 = 3 + \frac{1}{7} w_1 + \frac{1}{7} w_2 - \frac{2}{7} w_3$$

$$
\left.\begin{array}{l}
x_2 = -10 \cancel{+4x_1} - w_2 + w_3 \\
\qquad\quad 12 + \frac{4}{7}w_1 + \frac{4}{7}w_2 - \frac{8}{7}w_3
\end{array}\right] \Rightarrow x_2 = 2 + \frac{4}{7}w_1 - \frac{3}{7}w_2 - \frac{1}{7}w_3
$$

$$
\left.\begin{array}{l}
x_3 = 20 \cancel{-2x_1} \qquad - w_3 \\
\qquad -6 - \frac{2}{7}w_1 - \frac{2}{7}w_2 + \frac{4}{7}w_3
\end{array}\right] \Rightarrow x_3 = 14 - \frac{2}{7}w_1 - \frac{2}{7}w_2 - \frac{3}{7}w_3
$$

DICT:

$$
\begin{array}{rcl}
f = x_3 & = & 14 - \frac{2}{7}w_1 - \frac{2}{7}w_2 - \frac{3}{7}w_3 \\
\hline
x_1 & = & 3 + \frac{1}{7}w_1 + \frac{1}{7}w_2 - \frac{2}{7}w_3 \\
x_2 & = & 2 + \frac{4}{7}w_1 - \frac{3}{7}w_2 - \frac{1}{7}w_3 \\
x_3 & = & 14 - \frac{2}{7}w_1 - \frac{2}{7}w_2 - \frac{3}{7}w_3
\end{array}
$$

[CHECK: All bits of $\vec{x}^* = (3, 2, 14)$ and $\vec{y}^* = (\frac{2}{7}, \frac{2}{7}, \frac{3}{7})$ are in place!]

(d) New constraint $x_2 \le \frac{1}{2}$ could just add new slack to dict in (c):

$$
\left.\begin{array}{l}
w_4 = \frac{1}{2} \cancel{-x_2} \\
\qquad -2 - \frac{4}{7}x_1 + \frac{3}{7}w_2 + \frac{1}{7}w_3
\end{array}\right] \Rightarrow w_4 = -\frac{3}{2} - \frac{4}{7}W_1 + \frac{3}{7}w_2 + \frac{1}{7}w_3
$$

DUAL SIMPLEX OPPORTUNITY! $f + tw_4 = 14 - \frac{3}{2}t - (\frac{2}{7} + \frac{4}{7}t)w_1 - (\frac{2}{7} - \frac{3}{7}t)w_2 - (\frac{3}{7} - \frac{1}{7}t)w_3$

Now $t = 2/3$ selects w_2 to enter the basis. Pivoting will make

$$W_2 = \frac{3}{2}\left(\frac{7}{3}\right) = \frac{7}{2}, \quad w_4 = 0$$

and update

$$
\begin{aligned}
x_1 &= 3 + \frac{1}{7}\left(\frac{7}{2}\right) = 3.5 \\
x_2 &= 2 - \frac{3}{7}\left(\frac{7}{2}\right) = 0.5 \\
x_3 &= 14 - \frac{2}{7}\left(\frac{7}{2}\right) = 13.
\end{aligned}
$$

New maximizing point $(x_1^*, x_2^*) = (\frac{7}{2}, \frac{1}{2})$ will give $h^* = 13$.

[15] **3.** Archaeologists digging in the Nile valley have found the inscription below chiselled into a large block of limestone that has been buried for centuries.

$$\left[\begin{array}{lrl} \text{Maximize } f = & 3x_1 + 2x_2 + x_3 + 4x_4 & \\ \text{subject to} & x_1 + 3x_2 + x_3 + 3x_4 \leq & 40 \\ & -2x_1 - 3x_2 + x_3 + 3x_4 \leq & 8 \\ & x_2 \quad - x_4 \leq & -5 \\ & x_1, x_2, x_3, x_4 \geq & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{l} f = 95 - 3x_5 - 5x_7 - \\ \hline x_6 = \\ \\ \\ \\ \end{array}\right.$$

They can read the left part easily: it's a linear programming problem. But the block is broken, and only a fragment of the part on the right has survived. Assuming that the relic once showed a **correct optimal dictionary**, help the historians reconstruct the key missing ingredients.

You may assume that the ancients used x_5, x_6, x_7 as slack variables for the constraints in the order shown, but **you may not execute any simplex pivots.**

(a) Write the dual problem.

$$\begin{array}{llr} \min & g = 40y_1 + 8y_2 - 5y_3 & \\ \text{s.t.} & y_1 - 2y_2 & \geq 3 \\ & 3y_1 - 3y_2 + y_3 & \geq 2 \\ & y_1 + y_2 & \geq 1 \\ & 3y_1 + 3y_2 - y_3 & \geq 4 \\ & \vec{y} \geq \vec{0} \text{ in } \mathbb{R}^3 & \end{array}$$

(b) Find the optimal value in the dual problem.

$g_{MIN} = f_{MAX} = 95$, from the optimal dictionary fragment.

(c) Find an optimal input vector $\mathbf{y}^*$ for the dual problem.

Slack coeffs in optimal dict give $y_1^* = 3$, $y_2^* = 0$, $y_3^* = 5$.

(d) Find an optimal input vector $\mathbf{x}^*$ for the original problem.

Complementarity : $y_1^* > 0 \Rightarrow 0 = w_1 = 40 - x_1 - 3x_2 - x_3 - 3x_4$
$y_3^* > 0 \Rightarrow 0 = w_3 = -5 - x_2 + x_4$

Eval $z_1^* = 0$, $z_2^* = 12 > 0$, $z_3^* = 2 > 0$, $z_4^* = 0$.

Complementarity : $z_2^* > 0 \Rightarrow x_2 = 0$; $z_3 > 0 \Rightarrow x_3 = 0$.

System $\left.\begin{array}{r} x_1 + 3x_4 = 40 \\ x_4 = 5 \end{array}\right] \Rightarrow \left[\begin{array}{l} x_1 = 25 \\ x_4 = 5 \end{array}\right.$

ANSWER $\vec{x}^* = (25, 0, 0, 5)$.

(e) Identify the variables that are basic in the incomplete optimal dictionary.

x_6 is basic – it's written there!

x_1, x_4 must be basic – they are nonzero.

Dict can only have 3 basic vars, so these are the ones.

$$\mathcal{B} = \{1, 4, 6\}$$

(f) Reconstruct the incomplete objective row in the optimal final dictionary.

In RSM notation, $f = \vec{c}_{\mathcal{B}}^{\,T} B^{-1} \vec{b} - \left(\vec{c}_{\mathcal{B}}^{\,T} B^{-1} N - c_{\mathcal{N}}^T\right) \vec{x}_{\mathcal{N}}$.

Find $\vec{y}^{\,T} = c_{\mathcal{B}}^T B^{-1}$ from $\vec{y}^{\,T} B = c_{\mathcal{B}}^T$, i.e.,

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} x_1 & x_4 & x_6 \end{array} \\ \begin{bmatrix} 1 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & -1 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} x_1 & x_4 & x_6 \end{array} \\ \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} \end{array} \iff \begin{cases} y_1 - 2y_2 = 3 \\ 3y_1 + 3y_2 - y_3 = 4 \\ y_2 = 0 \end{cases}$$

Get $y_2 = 0,\ y_1 = 3,\ y_3 = 5,$ so now

$$z_{\mathcal{N}}^T = \vec{c}_{\mathcal{B}}^{\,T} B^{-1} N - \vec{c}_{\mathcal{N}}^{\,T} = \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 3 & 0 & 5 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_5 & x_7 \end{array} \\ \begin{bmatrix} 3 & 1 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array} - \begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_5 & x_7 \end{array} \\ \begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 14 & 3 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_5 & x_7 \end{array} \\ \begin{bmatrix} 12 & 2 & 3 & 5 \end{bmatrix} \end{array}$$

ANSWER: $f = 95 - 12x_2 - 2x_3 - 3x_5 - 5x_7$ if we sort it;

$f = 95 - 3x_5 - 5x_7 - 12x_2 - 2x_3$ as on rock in desert.

?order?

[15] 4. **Use the Revised Simplex Method (RSM)** to find all solutions (if any) for the problem

$$\max\left\{f = \mathbf{c}^T\mathbf{x} \,:\, A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \text{ in } \mathbb{R}^5\right\},$$

where

$$\mathbf{c}^T = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 2 & -1 & -1 & 0 \\ 1 & -4 & 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Start with the basis $\mathcal{B} = \{2, 5\}$, and follow the steps below.

(a) Find the current Basic Solution $\mathbf{x}$, and its objective value.

$\mathcal{B} = \{2,5\} \Rightarrow B = \left[\vec{a}^{(2)}\ \vec{a}^{(5)}\right] = \begin{bmatrix} 2 & 0 \\ -4 & 1 \end{bmatrix}$

$\vec{x}_{\mathcal{B}} = B^{-1}\vec{b} \iff B\vec{x}_{\mathcal{B}} = \vec{b} \iff \begin{bmatrix} 2 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \iff \begin{cases} 2x_2 = 4 \\ -4x_2 + x_5 = 2 \end{cases}$

Solve for $x_2 = 2$, $x_5 = 2 + 4x_2 = 10$.

Of course $\vec{x}_{\mathcal{N}} = \vec{0}$, so $\vec{x} = (0, 2, 0, 0, 10)$. Plug into $\underline{f = -4.}$

(b) Find the next entering variable (if there is one).

Solve for $\vec{y}$ in $\vec{y}^T B = \vec{c}_{\mathcal{B}}^T$, i.e., $\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \end{bmatrix}$:

$\left.\begin{array}{l} 2y_1 - 4y_2 = -2 \\ y_2 = 0 \end{array}\right\} \iff \begin{cases} y_1 = -1 \\ y_2 = 0. \end{cases}$

Let $\vec{z}_{\mathcal{N}}^T = \vec{y}^T N - \vec{c}_{\mathcal{N}}^T = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{array}{c} \begin{matrix} x_1 & x_3 & x_4 \end{matrix} \\ \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} - \begin{array}{c} \begin{matrix} x_1 & x_3 & x_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \begin{matrix} x_1 & x_3 & x_4 \end{matrix} \\ \begin{bmatrix} -2 & 0 & 1 \end{bmatrix} \end{array}.$

Negative entry selects $\boxed{E = 1}$ as the entering subscript.

(c) Find the next leaving variable (if there is one).

Solve for $\vec{d}$ in $B\vec{d} = \vec{a}^{(E)}$, i.e., $\begin{bmatrix} 2 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$:

$\left.\begin{array}{l} 2d_1 = 1 \\ -4d_1 + d_2 = 1 \end{array}\right\} \iff \begin{cases} d_1 = 1/2 \\ d_2 = 3. \end{cases}$

Let $\vec{x}_{\mathcal{B}}(t) = \vec{x}_{\mathcal{B}}(0) - t\vec{d} = \begin{bmatrix} 2 \\ 10 \end{bmatrix} - t\begin{bmatrix} 1/2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 - t/2 \\ 10 - 3t \end{bmatrix}.$

Smallest $t \geq 0$ to create a 0-element is $t = \frac{10}{3}$, which selects $\boxed{L = 5}$ as the leaving subscript.

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(d) Find the new basic feasible solution (BFS) after one pivot, and give its objective value.

When $t=\frac{10}{3}$ above, we will get $x_1=t=\frac{10}{3}$, $x_2=2-\frac{1}{2}\left(\frac{10}{3}\right)=\frac{1}{3}$ so
$$x=\left(\tfrac{10}{3},\tfrac{1}{3},0,0,0\right).$$

Substitute into objective: $f=\frac{10}{3}-\frac{2}{3}=\frac{8}{3}$.

(e) Is the problem now solved? If so, summarize your findings; if not, cycle back to step (b).
[Note: Complete at most one more RSM iteration. If this does not solve the problem, stop anyway.]

With $\mathcal{B}=\{1,2\}$ now,

- Solve $\begin{bmatrix} y_1 & y_2\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & -4\end{bmatrix}=\begin{bmatrix}1 & -2\end{bmatrix} \iff \begin{cases} y_1+y_2=1\\ 2y_1-4y_2=-2\end{cases}$

  for $y_1=1/3$, $y_2=2/3$.

- Build $\vec{z}_{\mathcal{N}}^T=\vec{y}^TN-\vec{c}_{\mathcal{N}}^T=\frac{1}{3}\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}-1 & -1 & 0\\ 1 & 0 & 1\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\end{bmatrix}$
$$=\frac{1}{3}\begin{bmatrix}1 & -1 & 2\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\end{bmatrix}=\begin{bmatrix}-\frac{2}{3} & -\frac{1}{3} & \frac{2}{3}\end{bmatrix}.$$

This problem is NOT SOLVED: negative entries in $\vec{z}_{\mathcal{N}}^T$ shows both vars x_3 and x_4 eligible to enter. Lets follow up with x_3.

- Solve $\begin{bmatrix}1 & 2\\ 1 & -4\end{bmatrix}\begin{bmatrix}d_1\\ d_2\end{bmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix} \iff \begin{cases} +d_1+2d_2=-1\\ d_1-4d_2=1\end{cases}$

  for $d_2=-1/3$, $d_1=-1/3$.

Imagine $x_3=t\geq 0$ and track $\vec{x}_{\mathcal{B}}(t)=\vec{x}_{\mathcal{B}}(0)-t\vec{d}$
$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}10/3\\ 1/3\end{bmatrix}-t\begin{bmatrix}-1/3\\ -1/3\end{bmatrix}=\begin{bmatrix}(10+t)/3\\ (1+t)/3\end{bmatrix}.$$

Here t can increase indefinitely without introducing infeasibility.

THIS PROBLEM IS UNBOUNDED. No maximizer exists!

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$\left[\ \vec{x}=\left(\frac{10+t}{3},\frac{1+t}{3},t,0,0\right)\right.$ is feasible, with $f(\vec{x}(t))=\frac{8+2t}{3}$, for each $\left. t\geq 0.\ \right]$

[15] **5.** Cleo and Rory play a zero-sum game. Cleo, the column player, chooses a probability vector $\mathbf{x}$ in $\mathbb{P}(3)$; Rory, the row player, chooses a probability vector $\mathbf{y}$ in $\mathbb{P}(4)$. Then Rory pays $\mathbf{y}^T A\mathbf{x}$ to Cleo, where

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 4 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

(a) Explain why the actions of both players can be predicted by studying a certain 3×2 matrix A_0 instead of the given 4×3 matrix A.

Row 3 is worse for Rory than either row 1 or row 4.
Either way, Rory will never play row 3.
With row 3 gone, Col 1 is worse for Cleo than Col 2.
So Cleo will never play Col 1.
The essential matrix of interest is

$$A_0 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 2 & 0 \end{bmatrix}.$$

(b) Set up a linear program to find Cleo's optimal probability vector $\mathbf{x}^*$. (You may use the simplified problem from part (a), or you may start from game as presented initially. Whatever you do, answer in terms of the given game setup, so that $\mathbf{x}^* \in \mathbb{P}(3)$.)

Cleo wants to maximize the minimum element of $A\vec{x}$.

ORIGINAL

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & v \le 2x_2 + x_3 \\ & v \le x_2 + 3x_3 \\ & v \le 3x_1 + 4x_2 + 2x_3 \\ & v \le x_1 + 2x_2 \\ & 1 = x_1 + x_2 + x_3 \\ & v \in \mathbb{R} \text{ free}; \ \vec{x} \ge \vec{0} \text{ in } \mathbb{R}^3 \end{aligned}$$

DOMINATED

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & v \le 2x_2 + x_3 \\ & v \le x_2 + 2x_3 \\ & v \le 2x_2 \\ & 1 = x_2 + x_3 \\ & v \in \mathbb{R} \text{ free}; \ (x_2, x_3) \ge (0,0) \text{ in } \mathbb{R}^2 \end{aligned}$$

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(c) Solve the LP in part (b). You may use any method, but you must explain your approach.

Sub $\boxed{x_3 = 1 - x_2}$ into the 'dominated' setup to get

$$\begin{array}{ll} \max & v \\ \text{st.} & v \le 2x_2 + 1 - x_2 \\ & v \le x_2 + 3 - 3x_2 \\ & v \le 2x_2 \\ & x_2 \le 1 \\ & v \in \mathbb{R} \text{ free; } x_2 \ge 0 \end{array} \iff \begin{array}{ll} \max & v \\ \text{st} & v - x_2 \le 1 \\ & v + 2x_2 \le 3 \\ & v - 2x_2 \le 0 \\ & x_2 \le 1 \\ & v \in \mathbb{R} \text{ free; } x_2 \ge 0 \end{array}$$

Let x_3 denote the slack for the last constraint, obtain dict

$$\begin{array}{l} f = v \\ \hline x_3 = 1 - x_2 \\ w_1 = 1 - v + x_2 \\ w_2 = 3 - v - 2x_2 \\ w_3 = -v + 2x_2 \end{array}$$

Pivot v into basis using $v = 2x_2 - w_3$, then drop it from dict.

$$\begin{array}{l} f = 2x_2 - w_3 \\ \hline x_3 = 1 - x_2 \\ w_1 = 1 - x_2 + w_3 \\ w_2 = 3 - 4x_2 + w_3 \end{array}$$

Enter $x_2 = \frac{3}{4} - \frac{1}{4}w_2 + \frac{1}{4}w_3$: $\left. f = \underbrace{\cancel{2x_2}}_{\frac{3}{2} - \frac{1}{2}w_2 + \frac{1}{2}w_3} - w_3 \right] \Rightarrow f = \frac{3}{2} - \frac{1}{2}w_2 - \frac{1}{2}w_3.$

So this will be optimal. In original vars, $\vec{x}^* = (0, \frac{3}{4}, \frac{1}{4})$.

(d) Find Rory's optimal probability vector $\mathbf{y}^*$. Check your answer; remember the instructions in part (b).

Slack coeffs in optimal objective row above suggest probabilities $\frac{1}{2}, \frac{1}{2}$ in final 2 slots of reduced vector $\vec{y}$. In original setup, $\vec{y}^* = (0, \frac{1}{2}, 0, \frac{1}{2})$. Check:

MaxElement$(\vec{y}^T A) = \frac{3}{2}$

$$\vec{y}^T A = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 4 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 3 & 3 \end{bmatrix}$$

Weak Duality: Equality here confirms mutual optimality

$$A\vec{x} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 4 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \left(\frac{1}{4}\right) = \frac{1}{4}\begin{bmatrix} 7 \\ 6 \\ 14 \\ 6 \end{bmatrix}$$

MinElement$(A\vec{x}) = \frac{3}{2}$.

[20] **6.** The hundredth anniversary of the Tour de France, the world's most famous cycling race, starts in just two days. Every racer is backed by a team of mechanics, trainers, and coaches who are optimizing every conceivable aspect of the next three weeks on the road. On long days, riders need to eat while they pedal. Our job is to plan a rider's snack pack—some mixture of nuts (x_1 grams), candy (x_2 grams), and "vitamins" (x_3 grams) that deliver the maximum energy boost. This energy, with units of kiloJoules (kJ), is

$$f = 4x_1 + 9x_2 + 5x_3.$$

Various limitations apply. The packet must not take up too much volume, so we insist on

$$x_1 + 2x_2 + x_3 \leq 6, \qquad \text{each term in deciliters, or d}\ell. \tag{1}$$

Biochemical considerations for the post-race medical review require

$$2x_1 + 5x_2 + 3x_3 \leq 15, \qquad \text{each term in milligrams, or mg.} \tag{2}$$

Finally, we don't want too many oxidants, so we require

$$3x_1 + 6x_2 + 4x_3 \leq 19, \qquad \text{each term in International Units, or IU.} \tag{3}$$

The numbers above are not exact; however, they have a convenient relationship to the matrix identity

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 6 & 4 \end{bmatrix} \iff B^{-1} = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix}.$$

(a) Write a standard linear programming problem that expresses the optimization problem above.

$$\begin{aligned} \max \quad & f = 4x_1 + 9x_2 + 5x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 6 \\ & 2x_1 + 5x_2 + 3x_3 \leq 15 \\ & 3x_1 + 6x_2 + 4x_3 \leq 19 \\ & \vec{x} \geq \vec{0} \text{ in } \mathbb{R}^3 \end{aligned}$$

(b) Write the dual problem for the problem in part (a).

$$\begin{aligned} \min \quad & g = 6y_1 + 15y_2 + 19y_3 \\ \text{s.t.} \quad & y_1 + 2y_2 + 3y_3 \geq 4 \\ & 2y_1 + 5y_2 + 6y_3 \geq 9 \\ & y_1 + 3y_2 + 4y_3 \geq 5 \\ & \vec{y} \geq \vec{0} \text{ in } \mathbb{R}^3 \end{aligned}$$

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(c) Confirm that an optimal snack mix involves positive quantities of nuts, candy, and vitamins. Identify the amounts involved.

Name the primal slack vars $(x_4, x_5, x_6) = (w_1, w_2, w_3)$.

The suggestion is that choosing $\mathcal{B} = \{1,2,3\}$ is optimal, with RSM setup

$$\vec{c}^T = \begin{bmatrix} 4 & 9 & 5 & 0 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 3 & 6 & 4 & 0 & 0 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 6 \\ 15 \\ 19 \end{bmatrix}$$

so $$B\vec{x}_{\mathcal{B}} = \vec{b} \iff \vec{x}_{\mathcal{B}} = B^{-1}\vec{b} = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 15 \\ 19 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

These choices make $(w_1, w_2, w_3) = (0,0,0)$, so they are feasible in (P).

Complementarity requires dual surplus vars $(z_1, z_2, z_3) = (0,0,0)$,

so we need $B^T\vec{y} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$ or $\vec{y} = (B^T)^{-1}\begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -3 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Dual-feasibility checks, so optimality above is confirmed.

(d) Is the optimal mixture unique? Discuss.

Work backwards from dual minimizer $\vec{y}^* = (2,1,0)$. What $\vec{x}$ solves (P)?

- Since $(z_1, z_2, z_3) = (0,0,0)$ no easy info about (x_1, x_2, x_3).
- Since $y_1 > 0, y_2 > 0$, we get $w_1 = 0, w_2 = 0$ for sure:

$$x_1 + 2x_2 + x_3 = 6$$
$$2x_1 + 5x_2 + 3x_3 = 15.$$

Sub $x_1 = 6 - 2x_2 - x_3$: $\quad (12 - 4x_2 - 2x_3) + 5x_2 + 3x_3 = 15$

$$\Rightarrow x_2 + x_3 = 3 \Rightarrow x_2 = 3 - x_3; \quad x_1 = 6 - (6 - 2x_3) - x_3 = x_3.$$

So for any real t, $(x_1, x_2, x_3) = (t, 3-t, t)$ obeys complementarity.

For primal feasibility, we need $t \geq 0$, $t \leq 3$, and

$$19 \geq 3t + 6(3-t) + 4t = 18 + t, \quad \text{or} \quad t \leq 1$$

Primal solution is NOT UNIQUE: full set of maximizers is $\vec{x} = (t, 3-t, t)$, $0 \leq t \leq 1$.

(e) A new company offers an edible synthetic product with an interesting profile: x_4 grams of this mystery substance would add $7x_4$ to the athlete's energy function, but occupy $2x_4$ dℓ of volume, add $2x_4$ mg of limited biochemicals, and contribute $3x_4$ IU of oxidants. Should we include some of this new product in an optimized snack pack?

In RSM notation, negative elements of $\vec{z}_{\mathcal{N}}^{\,T} = \vec{c}_{\mathcal{B}}^{\,T} B^{-1} N - \vec{c}_{\mathcal{N}}^{\,T}$ are desirable. We focus on the element of this vector associated with the new product. To get this, first let

$$\vec{y}^{\,T} = \vec{c}_{\mathcal{B}}^{\,T} B^{-1} = \begin{bmatrix} 4 & 9 & 5 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$$ (already saw this in (c)).

The coefficient of interest is

$$\vec{y}^{\,T} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 7 \end{bmatrix} = -1.$$

YES it is beneficial to include some of this product.

(f) Predict the change in the rider's energy gain, f_{MAX}, if the allocation vector $\mathrm{b} = (6, 15, 19)$ in lines (1)–(3) is changed to $\mathrm{b}' = (6 + p_1, 15 + p_2, 19 + p_3)$ for some perturbation vector $\mathrm{p} = (p_1, p_2, p_3)$ with small entries.

Typically, dual minimizer $\vec{y}^{\,*} = (2, 1, 0)$ reveals marginal payoff of available inputs. So we predict

$$\frac{\partial f_{max}}{\partial b_1} = y_1^* = 2, \quad \frac{\partial f_{max}}{\partial b_2} = y_2^* = 1, \quad \frac{\partial f_{max}}{\partial b_3} = y_3^* = 0,$$

and expect

$$\Delta f_{max} = 2\Delta b_1 + 1\Delta b_2 + 0\Delta b_3 = 2p_1 + p_2 .$$

(Degeneracy adds complications, but that level of detail is not expected here.)

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(e) A new company offers an edible synthetic product with an interesting profile: x_4 grams of this mystery substance would add $7x_4$ to the athlete's energy function, but occupy $2x_4$ dℓ of volume, add $2x_4$ mg of limited biochemicals, and contribute $3x_4$ IU of oxidants. Should we include some of this new product in an optimized snack pack?

ALTERNATIVE: New option changes (P)(D) above as follows:

(P) $\max\ f = 4x_1 + 9x_2 + 5x_3 + 7x_4$

s.t.
$$x_1 + 2x_2 + x_3 + 2x_4 \le 6$$
$$2x_1 + 5x_2 + 3x_3 + 2x_4 \le 15$$
$$3x_1 + 6x_2 + 4x_3 + 3x_4 \le 19$$
$$\vec{x} \ge \vec{0} \text{ in } \mathbb{R}^4$$

(D) $\min\ g = 6y_1 + 15y_2 + 19y_3$

s.t.
$$y_1 + 2y_2 + 3y_3 \ge 4$$
$$2y_1 + 5y_2 + 6y_3 \ge 9$$
$$y_1 + 3y_2 + 4y_3 \ge 5$$
$$2y_1 + 2y_2 + 3y_3 \ge 7$$
$$\vec{y} \ge \vec{0} \text{ in } \mathbb{R}^3$$

In view of (c), just check if $\vec{x}^* = (1,2,1,0)$ is optimal for this new (P). Complementarity again requires $(z_1, z_2, z_3) = (0,0,0)$ and gives $\vec{y}^* = (2,1,0)$ as before. But now dual-feasibility also concerns

$z_4^* = 2y_1^* + 2y_2^* + 3y_3^* - 7 = -1$. So leaving $x_4^* = 0$ is NOT OPTIMAL! Use some.

(f) Predict the change in the rider's energy gain, f_{MAX}, if the allocation vector $\mathbf{b} = (6, 15, 19)$ in lines (1)–(3) is changed to $\mathbf{b}' = (6 + p_1, 15 + p_2, 19 + p_3)$ for some perturbation vector $\mathbf{p} = (p_1, p_2, p_3)$ with small entries.

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(g) Calculate the actual change in the rider's energy gain, f_{MAX}, if the allocation vector $\mathrm{b} = (6, 15, 19)$ in lines (1)-(3) is changed to $\mathrm{b}' = (5, 14, 20)$. Compare the true value with the approximation suggested by part (f).

Rebuild lower dict using $\vec{x}_{\mathcal{B}} = B^{-1}\vec{b}' - B^{-1}N\vec{x}_{\mathcal{N}}$. Note $N = I$, so

$$B^{-1}b' = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 14 \\ 20 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= 2 - 2x_4 + 2x_5 - x_6 \\ x_2 &= -1 - x_4 - x_5 + x_6 \\ x_3 &= 5 + 3x_4 - x_6 \end{aligned}$$

Get objective row from $f = \vec{c}_{\mathcal{B}}^{\,T}B^{-1}\vec{b} - (\vec{c}_{\mathcal{B}}^{\,T}B^{-1}N - \vec{c}_{\mathcal{N}}^{\,T})x_{\mathcal{N}}$, or simply

$$f = 4x_1 + 9x_2 + 5x_3 = \left\{\begin{matrix} 8 & -8x_4 & +8x_5 & -4x_6 \\ -9 & -9x_4 & -9x_5 & +9x_6 \\ +25 & +15x_4 & & -5x_6 \end{matrix}\right\} = 24 - 2x_4 - x_5 - 0x_6.$$

Dict is INFEASIBLE, but DUAL-FEASIBLE, so start Dual Simplex Method:

$$f + tx_2 = (24 - t) - (2 + t)x_4 - (1 + t)x_5 - (0 - t)x_6.$$

Smallest $t \geq 0$ giving a 0-coeff is $t = 0$, selecting x_6 to enter, via

$$x_6 = 1 - x_2 - x_4 - x_5.$$

Look ahead: with nonbasics $x_2 = 0$, $x_4 = 0$, $x_5 = 0$, setting $x_6 = 1$ makes $x_1 = 1$, $x_3 = 4$. This will be optimal (since it's feasible); note $f = 24 - 0x_2 - 2x_4 - x_5$.

Comparison: Perturbation $\vec{p} = \vec{b}' - \vec{b} = (-1, -1, 1)$ suggests $(\Delta f)^{pred} = 2p_1 + p_2 = -3$

Actual $\Delta f = f_{MAX}^{(g)} - f_{MAX}^{(c)} = 24 - 27 = -3.$

Pretty close! (☺)

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[5] 7. Given a vector $\mathbf{c}$ in $\mathbb{R}^n$ and a matrix A of shape $m \times n$, prove that **one and only one** of the following two sets is nonempty:

$$S_1 \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \geq \mathbf{0} \text{ in } \mathbb{R}^m,\ \mathbf{c}^T\mathbf{x} < 0\},$$

$$S_2 \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y}^T A = \mathbf{c}^T,\ \mathbf{y} \geq \mathbf{0} \text{ in } \mathbb{R}^m\}.$$

Invent a (primal) LP based on the set S_1:

$$(P) \quad \max\left\{ f = (-\vec{c})^T\vec{x} : (-A)\vec{x} \leq \vec{0} \text{ in } \mathbb{R}^m,\ \vec{x} \in \mathbb{R}^n \text{ free}\right\}.$$

Obviously (P) is feasible, because $\vec{x} = \vec{0}$ in $\mathbb{R}^m$ satisfies the constraints.

- If $S_1 = \phi$ then every $\vec{x} \in \mathbb{R}^n$ with $A\vec{x} \geq \vec{0}$ in $\mathbb{R}^m$ must have $\vec{c}^T\vec{x} \geq 0$. Consequently $f(\vec{x}) \leq 0$ for each $\vec{x}$ feasible in (P), and this implies $\vec{x}^* = \vec{0}$ is a maximizer for (P).
- If $S_1 \neq \phi$ then there exists some vector $\tilde{x} \in S_1$. For each real $t \geq 0$, vector $t\tilde{x}$ is admissible in (P), where we have $f(t\tilde{x}) = t(-\vec{c})^T\tilde{x} = -t\underbrace{(\vec{c}^T\tilde{x})}_{<0} \to +\infty$ as $t \to +\infty$.
  So in this case (P) is unbounded.

The dual for (P) relies on the correspondence between free variables on one side with equality constraints on the other:

$$(D) \quad \min\left\{ g = \vec{0}^T\vec{y} : \vec{y}^T(-A) = (-\vec{c})^T,\ \vec{y} \geq \vec{0} \text{ in } \mathbb{R}^m\right\} = \min\{0 : \vec{y} \in S_2\}.$$

- If $S_1 = \phi$ then (P) has a maximizer, so (D) must have a minimizer. This means $S_2 \neq \phi$.
- If $S_1 \neq \phi$ then (P) is unbounded, so (D) must be infeasible. This means $S_2 = \phi$.